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CONTENTS

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A Class of Functions Harmonic within the Sphere.*

By H. E. Bray and G. C. Evans.

1. *Introduction.* Consider a function $u(M)$ harmonic at all points M of the interior Σ of the unit sphere S with center O. Let $F(e)$ represent a completely additive function of point sets on the surface S and let P and M represent points on the unit sphere and on the concentric sphere S_r or radius r, respectively. Our purpose is to investigate the situation in which $u(M)$ is given in terms of a Stieltjes integral by the formula

$$\text{(I)} \qquad u(M) = \frac{1}{4\pi} \int_S \frac{1-r^2}{\overline{MP}^3}\, dF(e),$$

or in terms of the Poisson integral

$$\text{(II)} \qquad u(M) = \frac{1}{4\pi} \int_S \frac{1-r^2}{\overline{MP}^3}\, f(P)\,dP,$$

of which the former is a generalization.

On account of equation (11) below, and its consequences, we shall deal with limits which will generally be given in terms of functions of curves. For these a simple class of curves will be sufficient, namely, the class of regular curves on the sphere. A regular closed curve is simple and consists of a finite number of regular arcs. Each regular arc has a continuously turning tangent at every interior point; it has a continuously turning forward (backward) tangent at the initial (terminal) point. Since we are dealing with closed curves on a sphere it will be necessary to distinguish between the two regions which they bound, and a curve will be completely specified only by its association with one of these regions as an interior region. This may be done, for example, by assigning a sense to the curve and putting the interior region on the left. A regular closed curve w possesses evidently the property, that, given a point P, the superficial metric density, at P, of the points of the region

* Report presented to the American Mathematical Society, April (1923). In the intervening time the authors have been able to simplify the treatment and reduce the conditions. See § 2.2 below.

interior to w—taken with respect to a family of concentric circles about P —is equal to the function

$$q(P, w) = 1,\ 0,\ \psi/2\pi$$

according as P is inside, outside, or on w; ψ being the interior angle from the forward to the backward tangent at P. It is this property which is of chief importance for our purposes. The quantity $q(P, w)$ regarded as a function of the point P will be seen to be measurable in the Borel sense.

1. 1. The most important of the regular curves is a *segment,* consisting of a curvilinear rectangle or polar cap, with reference to the usual angular coordinates on the sphere. We shall use the following notation:

N represents any fixed point of S whatever.

(r, θ, ϕ) represent spherico-polar coordinates of a point with regard to O, with N as north pole, θ as colatitude, ϕ as longitude.

s, or more explicitly $s(r;\ \theta_1, \theta_2;\ \phi_1, \phi_2)$ represents a figure on S_r called a segment. It is determined by the circles

$$\theta = \theta_1,\ \theta = \theta_2,\ \phi = \phi_1,\ \phi = \phi_2,$$

where

$$0 \leq \theta_1 < \theta_2 \leq \pi; \qquad 0 \leq \phi_1 < \phi_2 < 4\pi.$$

The figure is to be regarded as dividing the spherical surface into two regions, interior and exterior, mutually distinct; the interior points being defined as those whose coordinates (θ, ϕ) satisfy the following inequalities (a), (b), (c), (d), (e) according as the values of $\theta_1, \theta_2, \phi_1, \phi_2$, satisfy the respective relations (α), (β), (γ), (δ), (ϵ).

(a) $\theta_1 < \theta < \theta_2;\ \phi_1 < \phi < \phi_2$.	(α) $\theta_1 \neq 0,\ \theta_2 \neq \pi;\ \phi_2 - \phi_1 < 2\pi$.
(b) $\theta_1 < \theta < \theta_2$.	(β) $\theta_1 \neq 0,\ \theta_2 \neq \pi;\ \phi_2 - \phi_1 = 2\pi$.
(c) $\theta < \theta_2$.	(γ) $\theta_1 = 0,\ \theta_2 < \pi;\ \phi_2 - \phi_1 = 2\pi$.
(d) $\theta_1 < \theta$.	(δ) $\theta_1 > 0,\ \theta_2 = \pi;\ \phi_2 - \phi_1 = 2\pi$.
(e) $0 \leq \theta \leq \pi$.	(ϵ) $\theta_1 = 0,\ \theta_2 = \pi;\ \phi_2 - \phi_1 = 2\pi$.

These are the only cases considered in the definition of segments; (a) refers to a spherical rectangle, (b) to a zone, (c) and (d) to polar caps, and (e) to the whole sphere. It is to be noted, therefore, that a triangle or lune with a vertex at N or at its diametrically opposite point is not to be regarded as a segment. Of course, a point (θ, ϕ) on S is not to be regarded as distinct from the point $(\theta, \phi + 2\pi)$.

A *plurisegment* is a finite or denumerable set of segments no two of which have an interior point in common. If p_1 and p_2 are two plurisegments,

p_1 is said to be contained in p_2 if every segment of p_1 is contained in *a finite number* of segments of p_2. The two plurisegments are identical if each is contained in the other.*

1. 2. The basic instrument in the investigation to follow is the Stieltjes integral taken with respect to an additive bounded function of segments. The function $F(s)$ is an additive function of segments s if it is defined for every finite plurisegment and if when p_1 and p_2 are two finite plurisegments which have no interior point in common,

$$F(p_1) + F(p_2) = F(p_1 + p_2),$$

where $p_1 + p_2$ denotes the finite plurisegment consisting of the segments of p_1 and of p_2. The function $F(s)$ is said to be without point values if $F(s)$ becomes infinitely small with the diameter of s.

The positive variation function $P(s)$ for $F(s)$ is defined as the upper bound of $F(p)$ for all finite plurisegments p contained in s. This is again an additive function of segments and not negative; it is without point values if $F(s)$ is without point values.† Similarly, the function $N(s)$, defined as the analogous upper bound of $-F(p)$, is additive and of positive type, without point values if $F(s)$ is without point values. The same applies to the function $T(s)$, upper bound of $\Sigma \mid F(s_i) \mid$ where $p = \Sigma s_i$. In fact

$$F(s) = P(s) - N(s), \quad T(s) = P(s) + N(s).$$

The functions $N(s)$, $T(s)$ will be called respectively the negative and total variation functions of $F(s)$.

The extension of the definition of the function $P(s)$ is unique, if it is defined by means of the additive property for an infinite plurisegment $p = \Sigma s_i$. In fact, suppose $p = \Sigma s_i = \Sigma s'_i$ and $\Sigma P(s_i) \neq \Sigma P(s'_i)$, say

$$\sum P(s_i) = \sum P(s'_i) + 2\epsilon, \quad \epsilon > 0.$$

Then we can find a finite number n of segments s_i of p such that

$$\sum_1^n P(s_i) > \sum_1^\infty P(s'_i) + \epsilon.$$

But since $\sum_1^\infty s_i$ is identical with $\sum_1^\infty s'_i$, each s_i, and therefore $\sum_1^n s_i$, is contained in a finite number of segments s'_i. Hence, for some m,

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* A. J. Maria, "Functions of Plurisegments," *Transactions of the American Mathematical Society*, Vol. 28 (1926), pp. 448-471.

† The proof applies which is given in de la Vallée Poussin, *Intégrales de Lebesque*, Paris (1916), p. 99.

$$\sum_1^n P(s_i) \leq \sum_1^m P(s'_i);$$

which is in contradiction with the previous inequality.

It follows accordingly that the extension of the definition of $F(s)$, as an additive function, to infinite plurisegments p on S is unique:

$$F(p) = \sum_1^\infty F(s_i) \tag{1}$$

and the infinite series in question is absolutely convergent.

On the other hand, if $F(s)$ is a function of segments which is *completely additive,* that is, one for which $F(p)$ is uniquely defined, not infinite, for each plurisegment p on S by means of (1), it follows that $F(s)$ is a bounded additive function of segments, and further that

$$F(p_1 + p_2) = F(p_1) + F(p_2)$$

for infinite as well as finite plurisegments.*

In this résumé of the elementary properties of plurisegments all segments on S (with a given pole N) are considered. But it is important to notice that the theory also applies to certain subclasses of segments, e. g., to those of a *net.* We define a *net* H for a segment as follows. We form first a lattice H_n which is composed of a finite number of segments s'_{in} (with pole N), called meshes, each of diameter $< \delta_n$, and such that $s = \Sigma_i s'_{in}$. The system of lattices H_n corresponding to a set of numbers δ_n with $\lim \delta_n = 0$ constitutes the net H. In particular the net H may be defined for the whole sphere S, and we may thus deal with bounded additive functions $F(s')$ of segments formed from the meshes of the lattice.

1.21. The Riemann Stieltjes integrals.

$$\int_S h(P)\,dF(s_P) = \lim_{\delta=0} \sum_1^k h(P_i)F(s_i) \tag{2}$$

$$\int_S h(P)\,dF(s'_P) = \lim_{\delta=0} \sum_1^k h(P_i)F(s'_i) \tag{3}$$

where $\quad S = \sum_1^k s_i = \sum_1^k s'_i$, diam $s_i < \delta$, diam $s'_i < \delta$,

and P_i is a point of s_i or s'_i respectively, are uniquely defined, just as in the case of the usual Stieltjes integral, the function $F(s_P)$ being additive and

* A. J. Maria, *loc. cit.*, p. 449.

bounded as a function of segments on S (with pole N) and $F(s'_P)$ similarly as a function of segments of a net H.

The integrals (2) and (3) satisfy the properties (C), (A), (L), (M), of Daniell's S-integral.* Hence their definitions may be extended to any function $h(P)$ bounded on the sphere and measurable in the Borel sense, the fundamental interval in terms of which measure is defined being the segment; and if $h(P) = \lim_{k=\infty} h_k(P)$ where the $h_k(P)$ are bounded and have S-integrals, then the S-integral for $h(P)$ is the limit of that for $h_k(P)$. In particular, the function $q(P, w)$, already defined with reference to a regular curve w is measurable in the Borel sense, being obviously the limit of a sequence of continuous functions [see § 2].

If $F(s)$ is additive and bounded as a function of segments on S, the quantity

$$F_1(s) = \int_S q(P, s)\, dF(s_P) \tag{4}$$

is evidently also a bounded and additive function of all segments on S; this is true even if $F(s)$ is defined merely for the segments of a net H. If $F(s)$ satisfies the identity

$$F(s) = \int_S q(P, s)\, dF(s_P)$$

its possible discontinuities, in analogy to the case of a function of a single variable x, will be said to be *regular.*†

THEOREM. *Let $F(s')$ be bounded and additive on a net H and $F_1(s)$ be given by the definition*

$$F_1(s) = \int_S q(P, s)\, dF(s'_P). \tag{5}$$

Then if $h(P)$ is any bounded function, measurable Borel,

$$\int_S h(P)\, dF_1(s_P) = \int_S h(P)\, dF(s'_P) \tag{6}$$

and $F_1(s)$ has regular discontinuities.

To prove this, we observe first that if s is a segment of continuity for $F_1(s)$ then, clearly,

$$\int_S q(P, s)\, dF_1(s_P) = F_1(s) = \int_S q(P, s)\, dF(s'_P).$$

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* P. J. Daniell, "A General Form of Integral," *Annals of Mathematics*, Vol. 19 (1918), pp. 279-294.

† $F(s)$ is said to be continuous on s if it is continuous in the Volterra sense, as a function of curves. For a given $F(s)$ and pole N segments of discontinuity can be formed only from a certain denumerable infinity of meridians and parallels of latitude.

Let $h(P)$ be any continuous function. Consider a net on S all meshes of which are segments of continuity for $F(s)$, and let us choose in each mesh s_{in} of the lattice of order n a point P_{in} $(i=1, 2, \cdots, k_n)$, where k_n is the number of meshes in the lattice of order n. Then it is clear that the function

$$h_n(P) = \sum_{i=1}^{k_n} h(P_{in})\, q(P, s_{in})$$

will furnish an approximation to $h(P)$ which, in fact, approaches $h(P)$ uniformly as n becomes infinite. And since

$$\int_S h_n(P)\, dF_1(s_P) = \int_S h_n(P)\, dF(s'_P)$$

we have, when n becomes infinite,

$$\int_S h(P)\, dF_1(s_P) = \int_S h(P)\, dF(s'_P).$$

But by the theory of the S-integral, the identity holds also for $h(P)$ bounded and measurable in the Borel sense.

In particular

$$\int_S q(P, s)\, dF_1(s_P) = \int_S q(P, s)\, dF(s'_P),$$

i. e.,

$$F_1(s) = \int_S q(P, s)\, dF_1(s_P).$$

THEOREM. *If $F(s)$ has regular discontinuities, its positive negative and total variation functions have regular discontinuities on any segment.*

Since $F(s) = P(s) - N(s)$, we have

$$F(s) = \int_S q(P, s)\, dF(s_P) = \int_S q(P, s)\, dP(s_P) - \int_S q(P, s)\, dN(s_P)$$

$$F(p) = \int_S q(P, p)\, dP(s_P) - \int_S q(P, p)\, dN(s_P)$$

$$\leq \int_S q(P, p)\, dP(s_P) \leq \int_S q(P, s)\, dP(s_P),$$

for every p in s. Hence

$$P(s) \leq \int_S q(P, s)\, dP(s_P).$$

On the other hand if s_c represents the segment complementary to s with respect to S, we have similarly

$$P(s_c) \leq \int_S q(P, s_c)\, dP(s_P).$$

Hence by addition

$$\begin{aligned} P(S) &= P(s) + P(s_c) \\ &\leq \int_S [q(P, s) + q(P, s_c)]\, dP(s_P) = P(S) \end{aligned}$$

since $q(P,s) + q(P,s_c) \equiv 1$.

But this relation shows that only the equality sign is possible:

$$P(s) = \int_S q(P,s)\,dP(s_P)$$

and similarly

$$N(s) = \int_S q(P,s)\,dN(s_P)$$

and therefore

$$T(s) = \int_S q(P,s)\,dT(s_P).$$

1.22. If $F(s')$ is additive and bounded on a net H, the function

$$F_1(s) = \int_S q(P,s)\,dF(s'_P)$$

has been seen to be defined on all segments, with pole N, and to have regular discontinuities. To $F_1(s)$ there corresponds one and only one completely additive function of point sets $F(e)$ such that the equation

$$F_1(s) = F(e_s)$$

where e_s is the set of points interior to s, holds for every segment of continuity.* This function, in fact, is given for all sets e measurable Borel by the formula

$$F(e) = \int_S \phi(P,e)\,dF_1(s_P) = \int_S \phi(P,e)\,dF(s'_P) \tag{7}$$

where $\phi(P,e)$, the characteristic function for e, is defined by the relations

$$\begin{aligned} \phi(P,e) &= 1, \qquad P \text{ in } e, \\ &= 0, \qquad P \text{ in } Ce, \end{aligned} \tag{8}$$

and is therefore measurable Borel.

The Stieltjes integrals formed with respect to $F_1(s)$ and $F(e)$ are evidently the same for continuous integrands $h(P)$, since the approximating sums can be based on segments of continuity. Hence they are the same also for integrands $h(P)$ bounded and measurable Borel. Hence

$$\int_S h(P)\,dF(e_P) = \int_S h(P)\,dF_1(s_P) = \int_S h(P)\,dF(s'_P), \tag{9}$$

and in particular, for $h(P) = q(P,s)$

$$F_1(s) = \int_S q(P,s)\,dF(e_P). \tag{10}$$

We may speak of functions $F(e)$, $F_1(s)$, $F(s')$, etc. as corresponding if they satisfy (9) for all continuous functions (and therefore for all bounded

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* G. C. Evans, "Fundamental Points of Potential Theory," *Rice Institute Pamphlet*, Vol. 7 (1920), pp. 252-329. See p. 268.

functions, measurable Borel); given $F(s')$ there is therefore, by setting $h(P) = q(P, s)$ in (9), only one corresponding bounded additive function with regular discontinuities, and it is given by (5).

1. 3. We shall say that a set of additive functions $F_k(s)$ [or $F_k(e)$ or $F_k(s')$] are of *uniformly limited variation* if their total variation functions are bounded in their set. We shall say that a sequence of bounded additive functions $F_k(s)$ *approaches* $F(s)$ *on the net* H, if $\lim_{k=\infty} F_k(s_{in}) = F(s_{in})$ for every mesh s_{in} of the net.

It is enough to recall now the theorem of Helle-Bray: *

If the sequence $F_k(s)$, *additive and of uniformly limited variation for all* k, *approaches* $F(s)$, *additive and bounded, on the net* H *of* S, *then*

$$\lim_{k=\infty} \int_S h(P)\, dF_k(s_P) = \int_S h(P)\, dF(s_P), \tag{11}$$

where $h(P)$ *is a continuous function of* P. *Moreover if* $h_m(P)$ *is continuous, and approaches* $h(P)$ *uniformly,*

$$\lim_{\substack{k=\infty \\ m=\infty}} \int_S h_m(P)\, dF_k(s_P) = \int_S h(P)\, dF(s_P). \tag{12}$$

In particular, since H is a net for any plurisegment p composed of a finite number of meshes of the net, the theorem applies when S is replaced by p.

A similar theorem may evidently be stated in terms of additive functions of point sets, replacing $F_k(s)$ by $F_k(e)$, or $F(s)$ by $F(e)$, or both. If $\lim_{k=\infty} F_k(s)$ exists for every segment s, the limiting function $F(s)$ will necessarily be bounded and additive. The same is true with regard to $\lim_{k=\infty} F_k(e_s)$; the limit will always be a bounded additive function of segments s, but not necessarily a function of point sets.

2. *Necessary and sufficient conditions. Introductory Theorem. Given the function* u, *harmonic at every interior point of the unit sphere* S, *a necessary and sufficient condition that there exist on the surface of the sphere a bounded additive function of segments,* $F(s)$, *such that*

$$u(M) = \frac{1}{4\pi} \int_S \frac{1 - r^2}{\overline{MP}^3}\, dF(s_P) \tag{III}$$

is the following:

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* H. E. Bray, "Elementary Properties of the Stieltjes Integral," *Annals of Mathematics*, Vol. 20 (1919), pp. 177-186. See p. 180.

(a) *that there exist a sequence of values of* r, $r_1 < r_2 < r_3 \cdots$, *with* $\lim_{i=\infty} r_i = 1$, *such that for any segment* s, $F(r_i, s)$ *approaches a limiting value as* r_i *approaches* 1;

(b) *that* $F(r_i, s)$ *be of uniformly limited variation as a function of segments for all* i *where*

$$F(r,s) = \int_{\sigma(r,s)} u(M)\,dM,$$

and $\sigma(r,s)$ *represents the portion of* S_r *which is bounded by the projection from* O *of the segment* s *on* S.

The quantity $F(r,s)$ is a bounded additive function of segments s, for a given value of r. In order to prove the necessity of conditions (a) and (b) consider the quantity

$$p(P;r_i,s) = \frac{1}{4\pi}\int_{\sigma(r_i,s)} \frac{1-r_i^2}{\overline{MP}^3}\,dM,$$

which is continuous in P for every r_i and is bounded as a function of P, s and r. Moreover, as i tends to infinity, it approaches a definite limit for any P. In fact, P may be surrounded by an arbitrarily small circle, and the portion of $\sigma(1,s)$ outside that circle neglected. If the portion of $\sigma(1,s)$ inside the small circle is denoted by σ' and α and θ denote the angles of colatitude and longitude respectively, referred to OP, we have

$$\lim_{i=\infty} p(P;r_i,s) = \lim_{i=\infty} \frac{1}{4\pi}\int_{\sigma'(r_i,s)} \frac{(1-r_i)^2 r_i^2 \sin\alpha\, d\alpha\, d\theta}{(1+r_i^2-2r_i\cos\alpha)^{3/2}} \quad .$$

From this formula it follows immediately that

$$(13) \qquad \lim_{i=\infty} p(P;r_i,s) = q(P,s) = \begin{cases} 0 & (P \text{ outside } s) \\ 1 & (P \text{ inside } s) \\ \psi/2\pi & (P \text{ on } s), \end{cases}$$

where ψ denotes the angle from the forward to the backward tangent at any point of s.

But

$$F(r_i,s) = \frac{1}{4\pi}\int_{\sigma(r_i,s)} dM \int_S \frac{1-r_i^2}{\overline{MP}^3}\,dF(s_P)$$
$$= \int_S p(P;r_i,s)\,dF(s_P)$$

and since $p(P;r_i,s)$ is bounded, and $q(P,s)$ is measurable in the Borel sense on S, as a function of P

$$\lim_{i=\infty} F(r_i,s) = \int_S q(P,s)\,dF(s_P) = F_1(s)$$

where $F_1(s)$ is therefore the bounded additive function of segments with regular discontinuities, associated with $F(s)$ and identical with it except on segments for which the latter is discontinuous.

Condition (b) is established directly. In fact if the s_j are segments, we have, from the definition of the Stieltjes integral,

$$\sum_j |F(r_i, s_j)| \leq \frac{1}{4\pi}\int dM \int_S \frac{1-r_i^2}{\overline{MP}^3}\, dT(s_P)$$
$$\leq T(S)$$

where $T(s)$ is the total variation function of $F(s)$.

2.1. To show that the conditions (a), (b) are sufficient, consider now an arbitrary point M_0, fixed, interior to the spherical region bounded by S, and express $u(M_0)$, u being harmonic inside S, by means of the Poisson integral over S_i, the sphere of radius r_i; sufficiently large to contain M_0 as an interior point. Thus

$$u(M_0) = \frac{1}{4\pi r_i}\int_{S_i} \frac{r_i^2 - r_0^2}{\overline{M_0M}^3}\, u(M)\, dM \tag{14}$$
$$= \frac{1}{4\pi r_i}\int_{S_i} \frac{r_i^2 - r_0^2}{\overline{M_0M}^3}\, dF(r_i, s_M)$$
$$= \lim_{r_i=1} \frac{1}{4\pi r_i}\int_{S_i} \frac{r_i^2 - r_0^2}{\overline{M_0M}^3}\, dF(r_i, s_M)$$
$$= \frac{1}{4\pi}\int_S \frac{1-r_0^2}{\overline{M_0P}^3}\, dF_1(s_P)$$

by (12), since $F_1(s)$ is of limited variation.

Corollary. The conditions given are necessary and sufficient that $u(M)$ be given by (I), since the integrals (I) and (III) are identical when $F(s)$ and $F(e)$ are corresponding functions of segments and point sets respectively.

Corollary. For $u(M)$ given by (I) or (III) the conditions (a), (b) hold as r approaches 1 in any way whatever.

2.2. *Removal of condition* (*a*). We shall now prove that condition (b) implies condition (a), from which it will follow that (b) is a necessary and sufficient condition in order that $u(M)$ be given by (III). With the notation already defined for s, we have

$$F(r, s) = F(r;\, \theta_1, \theta_2;\, \phi_1, \phi_2) = \int_{\sigma(r,s)} u(r, \theta, \phi)\, d\sigma$$

$$T(r, s) = T(r;\, \theta_1, \theta_2;\, \phi_1, \phi_2) = \int_{\sigma(r,s)} |u(r, \theta, \phi)|\, d\sigma < K.$$

$F(r,s)$, $T(r,s)$ are therefore additive functions of segments s and functions of the parameter r; of uniformly limited variation in s as r ranges over the sequence $r_1 < r_2 < r_3 \cdots$. We wish to prove the following auxiliary theorem:

THEOREM. *If $F(r_i, s)$ is of uniformly limited variation for all values of i, then there exists a subsequence of values of r, $\{r_i'\}$, of the sequence $\{r_i\}$ and there exist two sets $E'(\theta)$, $E''(\phi)$ dense in the respective intervals $(0,\pi)$, $(0, 2\pi)$ such that*

$$\lim_{i=\infty} F(r'_i;\, \theta_1, \theta_2;\, \phi_1, \phi_2)$$

exists provided only that θ_1, θ_2, belong to $E'(\theta)$ and ϕ_1, ϕ_2 belong to $E''(\phi)$.

Suppose that $r' < r < 1$. Then, from Poisson's integral,

$$F(r', s) = \frac{1}{4\pi r} \int_{S_r} u(M)\, dM \int_{\sigma(r', s)} \frac{r^2 - r'^2}{\overline{MM'}^3}\, dM'$$

where M and M' represent points on the spheres of radii r, r' respectively. We can therefore write

$$F(r', s) = \int_{S_r} p(M; r, r'; s) u(M)\, dM$$

where p is a positive function, since $r > r'$, and is given by the formula

$$p(M;\, r,\, r';\, s) = \int_{\sigma(r', s)} \frac{r^2 - r'^2}{4\pi r \cdot \overline{MM'}^3}\, dM'.$$

We will consider various positions for the point M with regard to the boundary s.

(i) Suppose that M is inside s and that its minimum angular distance from s is β. Then

$$\begin{aligned} p(M; r, r'; s) &> \frac{(r^2 - r'^2)r'^2}{4\pi r} \int_0^{2\pi} d\gamma \int_0^{\beta} \frac{\sin \alpha\, d\alpha}{(r^2 + r'^2 - 2rr' \cos \alpha)^{3/2}} \\ &= \frac{r'(r + r')}{2r^2} - \frac{r'(r^2 - r'^2)}{2r^2(r^2 + r'^2 - 2rr' \cos \beta)^{1/2}} . \end{aligned}$$

(ii) Suppose that M is outside s and that its minimum angular distance from the boundary s is β. Then

$$\begin{aligned} 0 < p(M; r, r'; s) &< \frac{(r^2 - r'^2)r'^2}{4\pi r} \int_0^{2\pi} d\gamma \int_{\beta}^{\pi} \frac{\sin \alpha\, d\alpha}{(r^2 + r'^2 - 2rr' \cos \alpha)^{3/2}} \\ &= \frac{(r^2 - r'^2)r'}{2r^2} \left[\frac{1}{(r^2 + r'^2 - 2rr' \cos \beta)^{1/2}} - \frac{1}{r + r'} \right] \end{aligned}$$

(iii) Whatever may be the position of M with respect to s

$$0 < p(M; r, r'; s) \leq \frac{(r^2 - r'^2)r'}{2r^2} \left[\frac{1}{r - r'} - \frac{1}{r + r'}\right] = \frac{r'^2}{r^2} < 1.$$

We see from the above inequalities, that, given the increasing sequence $\{r_k\}$ it is possible to associate with any integer n a quantity δ_n with $\lim_{n=\infty} \delta_n = 0$ such that if the point $M(\theta, \phi)$ is inside the segment

$$\bar{s}(\theta_1 + \delta_n, \ \theta_2 - \delta_n; \ \phi_1 + \delta_n, \ \phi_2 - \delta_n)$$

or outside the segment

$$\bar{\bar{s}}\,(\theta_1 - \delta_n, \ \theta_2 + \delta_n; \phi_1 - \delta_n, \ \phi_2 + \delta_n)$$

then, for all $j > i > n$

$$| \, p(M; \ r_j, \ r_i; \ s) - q(M, s) \, | < \delta_n,$$

where $q(M, s)$ represents the function which is equal to 1, 0, $\psi/2\pi$ according as M is inside, outside, or on s.

Consequently,

$$\begin{aligned}
&| \, F(r_i; \theta_1, \theta_2; \phi_1, \phi_2) - F(r_j; \theta_1, \theta_2; \phi_1, \phi_2) \, | \\
&= \left| \int_{S_1} \{q(M, s) - p(M; \ r_j, \ r_i; \ s)\} u(M) \, dM \right| \\
&\leq \int_{S_1} | \, q(M, s) - p(M; \ r_j, \ r_i; \ s) \, | \cdot | \, u(M) \, | \, dM \\
&\leq \delta_n K + T(r_j; \ \theta_1 - \delta_n, \ \theta_1 + \delta_n; \ c, \ d) \\
&\qquad + T(r_j; \ \theta_2 - \delta_n, \ \theta_2 + \delta_n; \ c, \ d) \\
&\qquad + T(r_j; \ a, \ b; \ \phi_1 - \delta_n, \ \phi_1 + \delta_n) \\
&\qquad + T(r_j; \ a, \ b; \ \phi_2 - \delta_n, \ \phi_2 + \delta_n)
\end{aligned}$$

where the numbers $\theta_1 \pm \delta_n$, $\theta_2 \pm \delta_n$ lie between a and b, and $\phi_1 \pm \delta_n$, $\phi_2 \pm \delta_n$ lie between c and d; $0 \leq a < b \leq \pi$, $0 \leq c < d \leq 2\pi$. We now show that, given $\epsilon > 0$, n can be chosen so large that the integral

$$\int_R | \, F(r_i; \theta_1, \theta_2; \phi_1, \phi_2) - F(r_j; \theta_1, \theta_2; \phi_1, \phi_2) \, | \, dR < \epsilon \qquad (n < i < j)$$

where R is the region $[0 \leq \theta_1 \leq \pi, 0 \leq \theta_2 \leq \pi; \ 0 \leq \phi_1 \leq 2\pi, 0 \leq \phi_2 \leq 2\pi]$.

We first choose a positive number η so small that for all values of i, j

$$\int_{R-R_\eta} | \, F(r_i; \theta_1, \theta_2; \phi_1, \phi_2) - F(r_j; \theta_1, \theta_2; \phi_1, \phi_2) \, | \, dR < \epsilon/2$$

where R_η represents the region

$$[\eta \leqq \theta_1 \leqq \pi - \eta,\ \eta \leqq \theta_2 \leqq \pi - \eta;\ \ \eta \leqq \phi_1 \leqq 2\pi - \eta,\ \eta \leqq \phi_2 \leqq 2\pi - \eta].$$

This can be done because F, by hypothesis, is bounded. The η being now fixed, we consider only values of n for which $\delta_n < \eta$. For such values of n, it follows, that if $j > i > n$ and if $(\theta_1, \theta_2; \phi_1, \phi_2)$ lies in R_η

$$\begin{aligned}
&| F(r_i; \theta_1, \theta_2; \phi_1, \phi_2) - F(r_j; \theta_1, \theta_2; \phi_1, \phi_2) | \\
&< \delta_n K + T(r_j; \theta_1 - \delta_n, \theta_1 + \delta_n; 0, 2\pi) \\
&\qquad + T(r_j; \theta_2 - \delta_n, \theta_2 + \delta_n; 0, 2\pi) \\
&\qquad + T(r_j; 0, \pi; \phi_1 - \delta_n, \phi_1 + \delta_n) \\
&\qquad + T(r_j; 0, \pi; \phi_2 - \delta_n, \phi_2 + \delta_n).
\end{aligned}$$

Consequently, we obtain, after integrating over R_η,

$$\begin{aligned}
(15)\qquad &\int_{R_\eta} | F(r_i; \theta_1, \theta_2; \phi_1, \phi_2) - F(r_j; \theta_1, \theta_2; \phi_1, \phi_2) | \, dR \\
&\leqq \delta_n K \cdot 4\pi^4 + 4\pi^3 \int_\eta^{\pi-\eta} T(r_j; \theta_1 - \delta_n, \theta_1 + \delta_n; 0, 2\pi) d\theta_1 + \cdots \\
&\leqq \delta_n K \cdot 4\pi^4 + 4\pi^3 \int_\eta^{\pi-\eta} \{T(r_j; 0, \theta_1 + \delta_n; 0, 2\pi) \\
&\qquad\qquad - T(r_j; 0, \theta_1 - \delta_n; 0, 2\pi)\} d\theta_1 + \cdots
\end{aligned}$$

where the terms omitted in the last expression are similar to the integral term written down. By changing the variable of integration this term can be written in the form

$$\begin{aligned}
&4\pi^3 \left\{ \int_{\eta+\delta_n}^{\pi-\eta+\delta_n} - \int_{\eta-\delta_n}^{\pi-\eta-\delta_n} T(r_j; 0, \theta; 0, 2\pi) d\theta \right\} \\
&= 4\pi^3 \left\{ \int_{\pi-\eta-\delta_n}^{\pi-\eta+\delta_n} - \int_{\eta-\delta_n}^{\eta+\delta_n} T(r_j; 0, \theta; 0, 2\pi) d\theta \right\} \\
&< 4\pi^3 \int_{\pi-\eta-\delta_n}^{\pi-\eta+\delta_n} T(r_j; 0, \theta; 0, 2\pi) d\theta \\
&< 4\pi^3 K \cdot 2\delta_i
\end{aligned}$$

since $T(r_j; 0, \theta; 0, 2\pi)$ is a non-decreasing function of θ, less than K. By treating the other terms of (15) in a similar manner we arrive at the result

$$\int_{R\eta} | F(r_i; \theta_1, \theta_2; \phi_1, \phi_2) - F(r_j; \theta_1, \theta_2; \phi_1, \phi_2) | \, dR$$

$$< \delta_n K [4\pi^4 + 8\pi^3 + 8\pi^3 + 4\pi^3 + 4\pi^3] = \delta_n \cdot 4\pi^3 K(\pi + 6) \qquad (j > i > n).$$

We now choose n so large as to make this expression less than $\epsilon/2$. From this it follows that the integral extended over the whole of R will be less than ϵ for a sufficiently large value of n; that is to say, *the sequence of functions*

$$F(r_i; \theta_1, \theta_2; \phi_1, \phi_2)$$

converges in the mean (*of order* 1) *in the region* R.

We can now state the following auxiliary results which are implied by convergence in the mean, of order 1:

(A) *There exists a subsequence* $\{\bar{r}_i\}$ *of* $\{r_i\}$ *and a function* $\Phi(\theta_1, \theta_2; \phi_1, \phi_2)$ *defined at nearly every point* $(\theta_1, \theta_2; \phi_1, \phi_2)$ *of* R, *such that*

$$\lim_{i=\infty} F(\bar{r}_i; \theta_1, \theta_2; \phi_1, \phi_2) = \Phi(\theta_1, \theta_2; \phi_1, \phi_2)$$

except for a set of points, in R, *of measure zero.*

(B) *There exists a subsequence* $\{r'_i\}$ *of* $\{\bar{r}_i\}$ *and a function of* θ *denoted by* $\Phi(0, \theta; 0, 2\pi)$, *defined at nearly every point* θ *of the interval* $0 \leq \theta \leq \pi$, *such that*

$$\lim_{i=\infty} F(r'_i; 0, \theta; 0, 2\pi) = \Phi(0, \theta; 0, 2\pi),$$

except for a possible set of values of θ *of linear measure zero.*

The proof of (B) is similar to that already given, of (A), but it is less complicated and may therefore be omitted. But (A) is valid, of course, if r takes on the values in the subsequence $\{r'_i\}$ of $\{\bar{r}_i\}$; hence there exists a single sequence $\{r'_i\}$ for which both (A) and (B) are valid.

Let $E_1(\theta)$ represent the set of values of θ, of measure π, for which $\lim F(r'_i; 0, \theta; 0, 2\pi) = \Phi(0, \theta; 0, 2\pi)$. It follows from (A) that there exists at least one number a in $E_1(\theta)$ and a certain subset $E_2(\theta)$ of $E_1(\theta)$ of measure π, depending on a, such that the set of points (ϕ_1, ϕ_2) for which $\lim F(r'_i; a, \theta_2; \phi_1, \phi_2)$ does not exist is of superficial measure zero, if θ_2 is a number of the set $E_2(\theta)$. And since

$$F(r'_i; \theta_1, \theta_2; \phi_1, \phi_2) = F(r'_i; a, \theta_2; \phi_1, \phi_2) - F(r'_i; a, \theta_1; \phi_1, \phi_2),$$

it follows that if θ_1, θ_2 are any two numbers in $E_2(\theta)$, then there can be at most a set of points (ϕ_1, ϕ_2) of two-dimensional measure zero for which $\lim F(r'_i; \theta_1, \theta_2; \phi_1, \phi_1)$ does not exist.

Let us now choose a denumerable subset $E'(\theta)$ of $E_2(\theta)$ dense in the interval $(0, 2\pi)$ and let us exclude all points (ϕ_1, ϕ_2) for which $\lim F(r'_i; \theta_1, \theta_2; \phi_1, \phi_2)$ does not exist; allowing θ_1, θ_2 to take on every pair of numbers in $E'(\theta)$. We thus exclude a denumerable aggregate of sets of points $\phi_1, \phi_2)$ each of measure zero. The points (ϕ_1, ϕ_2) remaining constitute a set $E_1(\phi_1, \phi_2)$ of measure $4\pi^2$ in the region $0 \leq \phi_1 \leq 2\pi$; $0 \leq \phi_2 \leq 2\pi$. If now θ_1, θ_2 are in $E'(\theta)$ and if (ϕ_1, ϕ_2) is in $E_1(\phi_1, \phi_2)$ then $\lim F(r'_i; \theta_1, \theta_2; \phi_1, \phi_2)$ exists.

There exists a number b, $0 < b < 2\pi$, and a set $E''(\phi)$ such that meas. $E''(\phi) = 2\pi$ and such that $\lim F(r_i'; \theta_1, \theta_2; b, \phi_2)$ exists if ϕ_2 is any number in $E''(\phi)$. And, since

$$F(r'_i; \theta_1, \theta_2; \phi_1, \phi_2) = F(r'_i; \theta_1, \theta_2; b, \phi_2) - F(r'_i; \theta_1, \theta_2; b, \phi_1),$$

it follows that if θ_1, θ_2 are in $E'(\theta)$ and if ϕ_1, ϕ_2 are in $E''(\phi)$

$$\lim_{i=\infty} F(r'_i; \theta_1, \theta_2; \phi_1, \phi_2) = \Phi(\theta_1, \theta_2; \phi_1, \phi_2) = \Phi(s')$$

exists. The auxiliary theorem is thus proved.

We are now enabled to complete the proof of the statement that condition (b) implies condition (a), that is, that $\lim F(r'_i, s)$ exists for every segment s. First, we have

$$\begin{aligned} F(r'_i, s) &= \lim_{j=\infty} \int_{S_j} p(M; r'_j, r'_i; s)\, dF(r_j', s'_M) \\ &= \int_S p(P; 1, r'_i; s)\, d\Phi(s'_P) \end{aligned}$$

where the integrals are formed with respect to any net H of segments s' determined exclusively by the circles $\theta = \text{const.}$ in $E'(\theta)$, $\phi = \text{const.}$ in $E''(\phi)$. For $p(M; r_j, r_i; s)$ approaches $p(P; 1, r_i; s)$ uniformly, and $F(r_j, s')$ approaches $\Phi(s')$ on every segment s' in H. Evidently $\Phi(s')$ is a bounded additive function of segments on the net H. Hence

$$\begin{aligned} \lim_{i=\infty} F(r'_i, s) &= \lim_{i=\infty} \int_S p(P; 1, r'_i; s)\, d\Phi(s'_P) \\ &= \int_S q(P, s)\, d\Phi(s'_P) \end{aligned}$$

by the definition of the generalized Stieltjes integral. For the function $q(P, s)$ is again the limit function of the bounded sequence of continuous functions of P, $p(P; 1, r'_i; s)$. Thus condition (a) is satisfied.

We are thus able to state the following theorem, which summarizes the results of these sections.

THEOREM. *A necessary and sufficient condition that $u(M)$; harmonic in Σ, be given by the formula* (I),

$$u(M) = \frac{1}{4\pi} \int_S \frac{1-r^2}{\overline{MP}^3} dF(e), \tag{I}$$

where $F(e)$ is a completely additive function of point sets, or by (III)

$$u(M) = \frac{1}{4\pi} \int_S \frac{1-r^2}{\overline{MP}^3} dF(s) \tag{III}$$

where $F(s)$ is a bounded and additive function of segments is that

$$\int_{S_r} | u(M) | dM \tag{b}$$

remain bounded as r approaches 1, over a denumerable sequence of values, or, in fact, in any way.

The function $F(r,s)$ has a limit as r approaches 1 for every segment s, and

$$\begin{aligned} \lim_{r=1} F(r,s) &= \lim_{r=1} \int_S p(P;\, 1,\, r;\, s) dF(s) \\ &= \int_S q(P,s) dF(s) = F_1(s) \end{aligned} \tag{16}$$

is identical with $F(s)$ on all segments of continuity and has regular discontinuities; the function $F(e)$ is the corresponding completely additive function of point sets.

In fact, the above relations are true if $F_1(s)$ and $F(e)$ are functions corresponding to $F(s)$ as described in § 1. 22. On the other hand, if $F_2(e)$ were a completely additive function of point sets, different from $F(e)$, i. e., not corresponding to $F(s)$, the corresponding bounded additive function of segments $F_2(s)$ with regular discontinuities would differ from $F_1(s)$ on some segment and relation (16) would fail.

2. 3. The class of functions given by (I) or (III) within the unit sphere may also be characterized in another way.

THEOREM. *A necessary and sufficient condition that $u(M)$ be given by* (I) *or* (III) *within the unit sphere is that it be the difference of two functions*

$$u(M) = u_1(M) - u_2(M)$$

which are harmonic and not negative within the unit sphere.

The condition is evidently necessary, since such functions $u_1(M)$, $u_2(M)$

may be obtained by substituting for $F(s)$ in (III) the positive and negative variation functions, respectively, of $F(s)$. In order to prove the sufficiency we note that

$$\int_{S_r} | u(M) | \, dM \leq \int_{S_r} u_1(M) dm + \int_{S_r} u_2(M) dM$$
$$= 4\pi r^2 [u_1(O) + u_2(O)] < 4\pi [u_1(O) + u_2(O)].$$

Hence condition (b) is satisfied.

2.4. In § 1.21 the reader was allowed to assume that the pole for the systems of segments remained invariant, although the formulae there given provide convenient means for just such changes of coordinates. Rather than complete the demonstrations in this more general sense let us turn at once to the general class of regular curves on S.

We define a *regular net* G on S, for a regular curve w as a system of lattices G_n corresponding to positive numbers δ_n successively decreasing and approaching 0 as a limit. Each lattice represents a partition of w and its interior into a finite number of cells w_{in} each of which is a regular curve of diameter $< \delta_n$. Given a net G for a regular curve w it may obviously be extended throughout the complement w_c of w so as to become a net for S itself.

Let $F(s')$ be additive and bounded on a net H of segments, and define

$$F_1(w) = \int_S q(P, w) dF(s'_P) \tag{17}$$

for any regular curve w on S. Then $F_1(w)$ is a function of regular curves, additive and bounded on S. Moreover we can integrate by means of a Riemann sum with respect to $F_1(w)$ any function $h(P)$ if it is continuous on S, and extend the definition of the integral as an S-integral for any $h(P)$ bounded and measurable Borel. Take first $h(P)$ to be continuous.

Divide S into portions bounded by regular curves w_i of diameter $< \delta$. Write

$$h_\delta(P) = \sum_i h(P_i) q(P, w_i),$$

P_i a point of w_i. Then

$$\sum_i h(P_i) F_1(w_i) = \int_S h_\delta(P) dF(s'_P)$$

by definition of $F_1(w_i)$. But the limit of the right hand side is unique as δ approaches 0, since $\lim\limits_{\delta=0} h_\delta(P) = h(P)$ uniformly. Hence the Riemann-Stieltjes sum in the left-hand member has a unique limit; it also satisfies

2

the postulates (C) (A) (L) (M) of the S-integrals.* It follows therefore that

$$\int_S h(P)\,dF_1(w_P) = \int_S h(P)\,dF(s'_P) \tag{18}$$

for all functions $h(P)$ measurable Borel and bounded.

In particular we may take $h(P) = q(P, w)$ so that

$$\int_S q(P, w)\,dF_1(w_P) = F_1(w) \tag{19}$$

and the discontinuities of $F_1(w)$ may be said to be *regular.*

Let $P_1(w)$ be the upper bound of $F_1(p_w)$ for all finite pluricells p_w composed of arbitrary regular curves w_i contained in w, and let $N_1(w)$, $T_1(w)$ be defined accordingly. Let $P_1(s')$, $N_1(s')$, $T_1(s')$ be the functions similarly formed on H. We have the following theorem.

THEOREM. *The function $P_1(w)$ is given by the formula*

$$P_1(w) = \int_S q(P, w)\,dP_1(s'_P) \tag{20}$$

and is a bounded additive function of regular curves with regular discontinuities. Corresponding relations hold for $N_1(w)$ and $T_1(w)$.

It is necessary merely to prove (20), since the rest of the theorem is an immediate consequence of that relation. Let G be a net of regular curves w'_{in} for both w and w_c and form $P'(w')$ and $N'(w')$ as upper bounds, of $F_1(p_{w'})$ and $-F_1(p_{w'})$ respectively, on G, for curves w' which are finite pluricells on the net. The analysis of §§ 1.2, 1.21 shows that these functions are bounded and additive for curves w' and satisfy the relations

$$\begin{aligned} &F(w') = P'(w') - N'(w') \\ &P'(w') = \int_S q(P, w')\,dP'(w'_P) \\ &P'(w) + P'(w_c) = P'(S) \leq P_1(w) + P_1(w_c). \end{aligned} \tag{21}$$

We have, however, from (19), if p is a finite plurisegment in H,

$$\begin{aligned} F_1(p) &= \int_S q(P, p)\,dF_1(w'_P) \\ &= \int_S q(P, p)\,dP'(w'_P) - \int_S q(P, p)\,dN'(w'_P) \\ &\leq \int_S q(P, p)\,dP'(w'_P) \leq \int_S q(P, s')\,dP'(w'_P) \end{aligned}$$

when the plurisegment p is contained in the segment s' of H. Hence

$$P_1(s') \leq \int_S q(P, s')\,dP'(w'_P);$$

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* P. J. Daniell, *loc. cit.*

also

$$P_1(s'_c) \leq \int_S q(P, s'_c)\, dP'(w'_P),$$

and therefore

$$P_1(S) = P_1(s') + P_1(s'_c) \leq \int_S [q(P, s') + q(P, s'_c)]\, dP'(w'_P)$$

$$(22) \qquad P_1(S) \leq P'(S).$$

On the other hand

$$\begin{aligned} F_1(p_w) &= \int_S q(P, p_w)\, dF_1(s'_P) \\ &= \int_S q(P, p_w)\, dP_1(s'_P) - \int_S q(P, p_w)\, dN_1(s'_P) \\ &\leq \int_S q(P, p_w)\, dP_1(s'_P) \leq \int_S q(P, w)\, dP_1(s'_P). \end{aligned}$$

Hence

$$P_1(w) \leq \int_S q(P, w)\, dP_1(s'_P),$$

$$P_1(w_c) \leq \int_S q(P, w_c)\, dP_1(s'_P).$$

Unless therefore (20) is satisfied we have

$$P_1(w) + P_1(w_c) < P_1(S).$$

But this equation is not compatible with (22) and the third of equations (21). Hence (20) must be valid.

Incidentally, *we have the fact that*

$$(23) \qquad P_1(w) = P'(w).$$

Since w is itself a pluricell of G.

2. 5. As an application of § 2. 4 we have the following theorem:

THEOREM. *If $u(M)$ is given by* (III), *and $F_1(w)$ is the additive and bounded function of regular curves w associated with $F(s)$ by the equation*

$$(24) \qquad F_1(w) = \int_S q(P, w)\, dF(s_P)$$

then

$$(25) \qquad \lim_{r=1} \left(F(r, w) = \int_{\sigma(r,w)} u(M)\, dM \right) = F_1(w) = P_1(w) - N_1(w)$$

$$\lim_{r=1} \left(T(r, w) = \int_{\sigma(r,w)} | u(M) |\, dM \right) = T_1(w) = P_1(w) + N_1(w)$$

where $\sigma(r, w)$ is the projection of the region bounded by w on the sphere of radius r.

In fact

$$F(r,w) = \lim_{r'=1}\int_{S_{r'}} p(M;r',r;w)\,dF(r',s)$$
$$= \int_S p(P;1,r;w)\,dF(s),$$

by (12), as in § 2. 2. And therefore

$$\lim_{r=1} F(r,w) = \int_S q(P,w)\,dF(s) = F_1(w).$$

Since $p(P;1,r;w)$ remains bounded as r tends to 1. Thus the first of the relations (25) is established.

For the second relation, we obtain directly from the Poisson integral the inequality

$$\int_{\sigma(r,w)} |u(M)|\,dM \leq \int_S p(P;1,r;w)\,dP_1(s) + \int_S p(P;1,r;w)\,dN_1(s),$$

and therefore

$$\lim_{r=1}\int_{\sigma(r,w)} |u(M)|\,dM \leq \int_S q(P,w)\,dP_1(s) + \int_S q(P,w)\,dN_1(s)$$

or

$$\lim_{r'=1} T(r,w) \leq P_1(w) + N_1(w).$$

On the other hand, since $P_1(w)$ by (23) is the upper bound $P'(w)$ of $F_1(p_w)$ for all finite pluricells of a net G for w, contained in w, we can find such a finite pluricell p_w such that

$$F_1(p_w) > P_1(w) - \epsilon/4, \qquad \epsilon \text{ arbitrary, positive;}$$

and if we represent by p'_w the finite pluricell of G complementary to p_w with respect to w, we have

$$-F_1(p'_w) > N_1(w) - \epsilon/4.$$

We may now choose r near enough to 1 so that

$$F(r,p_w) > F_1(p_w) - \epsilon/4$$
$$-F(r,p'_w) > -F_1(p'_w) - \epsilon/4.$$

By combining these results, we obtain,

$$F(r,p_w) - F(r,p'_w) > P_1(w) + N_1(w) - \epsilon.$$

Hence

$$T(r,w) \geqq F(r,p_w) - F(r,p'_w)$$
$$> P_1(w) + N_1(w) - \epsilon;$$

or

$$\lim_{r=1} T(r,w) \geq P_1(w) + N_1(w).$$

By comparing this inequality with the earlier one, the second of the relations (25) is verified.

3. *Behaviour of $u(M)$ in the neighborhood of the boundary.* An important theorem comes as a consequence of (I) or (III).

THEOREM. *Let A be a point of S at which $F(s)$ possesses a derivative $f(A)$,* and $M_1, M_2, \cdots$ be any sequence of points, with $\lim_{n=\infty} M_n = A$ such that, if r_i denotes OM_i and θ_i the angle between OM_i and OA, the quantity $\theta_i/(1-r_i)$ is bounded. Then, if $u(M)$ is given by* (III) *or* (I),

$$\lim_{i=\infty} u(M_i) = f(A).$$

If $F(s)$ has a unique derivative at A, the same is true of the corresponding function of segments with regular discontinuities. Hence we may assume that the discontinuities of $F(s)$ are regular.

We may write

$$F(s) = \sigma f(A) + g(s)$$

where $g(s)$ is a bounded additive function of segments with regular discontinuities, and has a unique derivative at A which is zero.

If a bounded additive function of segments $g(s)$ has a zero derivative at a point A, the same is true of its positive and negative variation functions. In fact, if p_n is such a regular family, we can choose p'_n in p_n so that

$$P(p_n) < F(p'_n) + 1/2^n \text{ (meas. } p_n),$$

and if we let p''_n be the complement of p'_n in p_n,

$$N(p_n) < -F(p''_n) + 1/2^n \text{ (meas. } p_n).$$

For every value of n, either p'_n or p''_n is of measure $\geq p_n/2$. Hence if we denote by p'_i those p'_n of measure $\geq p_n/2$, and by p''_j the p''_n corresponding to the remaining values of n, we shall have two regular families of plurisegments, of which the parameter of regularity is not more than twice that for p_n. Accordingly we have

$$\lim_{i=\infty} F(p'_i)/\text{meas. } p'_i = 0$$

* The derivative is taken with regard to any regular family of plurisegments. See A. J. Maria, *loc. cit.*, p. 459.

and therefore

$$\lim_{i=\infty} \frac{P(p_i)}{\text{meas. } p_i} \leq \lim_{i=\infty} \frac{F(p'_i)}{\text{meas. } p_i} + \lim_{i=\infty} 1/2^n = 0,$$

where i ranges over this certain sequence of values.

Similarly for the other sequence of values,

$$\lim_{j=\infty} N(p_j)/\text{meas. } p_j = 0$$

and therefore

$$\lim_{j=\infty} \frac{P(p_j)}{\text{meas. } p_j} = \lim_{j=\infty} \frac{F(p_j) + N(p_j)}{\text{meas. } p_j} = 0.$$

Hence for the whole sequence of values of n

$$\lim_{n=\infty} P(p_n)/\text{meas. } p_n = 0.$$

Accordingly we may assume for simplicity that $g(s)$ is of positive type. Since the discontinuities of $F(s)$, $g(s)$ are regular (see § 1. 21) the definition of these functions may be extended, by means of § 2. 4, to all regular curves on S, and the integral in (I) or (III) written with respect to $F(w)$. Moreover since $g(s)$, of positive type, has a zero derivative at A on a regular family of segments for which A is an interior point, the same is true at A for any regular family whatever of regular curves w.

Let now a small circle of latitude, C, be drawn upon S with OA as axis, and let α' be its colatitude; and let α' be chosen so small that for any family of circles (c) within C and containing A—this would be a regular family having the number 4 as parameter of regularity with respect to circles of center A—the value of F can be expressed as

$$F(c) = [f(A) + \eta(c)]\sigma \tag{26}$$

where σ is the area of c, and $0 \leq \eta(c) < \epsilon$ a quantity which approaches 0 with α'. Otherwise there would be a sequence of c's constituting a regular family on which the derivative of g would not be 0.

Since M_i is to approach A we may assume that $\theta_i < \alpha'/2$. Thus, if we denote $\alpha' - \theta_i$ by α_i, a circle c_i of colatitude α_i about OM_i as axis will contain A in its interior and will touch C internally. Now we have

$$\lim_{i=\infty} u(M_i) = f(A) + \lim_{i=\infty} (1/4\pi) \int_S \frac{1 - r_i^2}{\overline{M_i P}^3} \, dg(w)$$

$$=f(A)+\lim_{i=\infty}(1/4\pi)\int_S q(P,c_i)\ \frac{1-r_i^2}{\overline{M_iP}^3}\ dg(w),$$

since the circle c_i ultimately tends to C, and the distance from the projection of M_i on S to the complement $S-c_i$ of c_i exceeds a positive number which is independent of i.

Write now, for circles c with OM_i as axis and a as colatitude,

$$g(c)=h_i(a)$$

which is thus a function of a of limited variation, uniformly for all i. Moreover

$$h_i(a)=\eta_i(a)\cdot 2\pi(1-\cos a) \qquad \theta_i\leqq a\leqq a_i$$

$$h_i(a)\leq \eta_i(\theta_i)\cdot 2\pi(1-\cos\theta_i) \qquad 0\leq a\leq \theta_i,$$

in which $\eta_i(\theta_i)$ approaches zero as i becomes infinite. Now the integral may be evaluated with respect to any net on S. Hence by an integration by parts, we have for the limit to be considered one which does not exceed

$$\lim\,(1/4\pi)\int_0^{a_i}\frac{1-r_i^2}{1+r_i^2-2r_i\cos a}\ dh_i(a)$$

$$=\lim\,(1/4\pi)\left[\frac{(1-r_i^2)h_i(a)}{(1+r_i^2-2r_i\cos a)^{3/2}}\right]_0^{a_i}$$

$$+\lim\,(3/4\pi)\int_0^{a_i}\frac{(1-r_i^2)r_i\sin a\cdot h_i(a)}{(1+r_i^2-2r_i\cos a)^{5/2}}\ da$$

of which the first term is zero; for at $a=a_i$ the denominator has a fixed lower bound, and at $a=0$

$$(1/4\pi)\left[\frac{(1-r_i^2)h_i(a)}{(1+r_i^2-2r_i\cos a)^{3/2}}\right]_0\leq \eta_i(\theta_i)(1+r_i)\ \frac{\sin^2(\theta_i/2)}{(1-r)^2}\ .$$

The second term, when the integral is rewritten as $\int_0^{\theta_i}+\int_{\theta_i}^{a_i}$ may be handled directly, and is

$$\leq\lim\,(1/4\pi)\left[\frac{\eta_i(\theta_i)(1-\cos\theta_i)(1-r_i^2)}{(1+r_i^2-2r_i\cos a)^{3/2}}\right]_{a=0}^{a=\theta_i}$$

$$+\,(3/2)\,\epsilon\lim\int_0^{a_i}\frac{(1-r_i^2)r_i(1-\cos a)\sin a}{(1+r_i^2-2r_i\cos a)^{5/2}}\ da,$$

where for simplicity the lower limit of integration in the second term has

been replaced by 0. The first term has the limit 0, on account of the restriction on $\theta_i/(1-r_i)$ and the second may be valuated as

$$\epsilon \lim (1/4\pi) \left[\frac{(1-r_i)^3(1+r_i)}{(1+r_i^2-2r_i\cos\alpha)^{3/2}} - \frac{3(1-r_i)(1+r_i)}{(1+r_i^2-2r_i\cos\alpha)^{1/2}} \right]_{\alpha=0}^{\alpha=\alpha_i}$$

$$= \epsilon$$

Hence

$$\left| \lim_{i=\infty} u(M_i) - f(A) \right| < \epsilon$$

and

$$\lim_{i=\infty} u(M_i) = f(A).$$

In other words $u(M)$ *approaches* $f(A)$ *not only as* M *approaches* A *along a radius, but also as* M *approaches* A *along any path which may be contained in a circular cone of vertex* A *and axis* OA *with vertical semi-angle* $< \pi/2$ *.

4. *Poisson's integral and the Dirichlet problem for summable boundary values.* The functions $u(M)$ of the class we have been considering are evidently not determined by their boundary values $f(A)$. In fact it is the $F(e)$ which is arbitrary. That is, if $\lim_{r=1} F(r, s)$ is an arbitrary additive and bounded function of curves, with regular discontinuities, for regular curves on S, there is one and only one function $u(M)$ harmonic within S and subject to this boundary condition if $\int_{S_r} |u(M)|\, dM$ remains bounded. This is a *generalized* Dirichlet problem.

If however the lower net derivative † of $F(e)$, (which is $f(A)$ almost everywhere), does not anywhere have a positively infinite value (which it can have, in general on a point set of measure 0), or the upper net derivative a negatively infinite value, the $F(e)$ or $F(s)$ is absolutely continuous, the formula (I) or (III) degenerates into Poisson's Integral, and the function $u(M)$ is determined by assigning its boundary values, summable in the Lebesgue sense, almost everywhere on S.

THEOREM. *A necessary and sufficient condition in order that* $u(M)$, *harmonic in* Σ *be given by the Poisson integral*

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* A generalization of Fatou's well-known theorem for the circle; "Séries trigonométriques et séries de Taylor," *Acta Mathematica*, Vol. 35 (1906), pp. 335-400, see p. 345 and p. 357.

† The net derivative in the sense of de la Vallée Poussin, *loc. cit.*, p. 98.

$$\text{(II)}, \qquad u(M) = (1/4\pi)\int_S \frac{(1-r^2)f(P)}{\overline{MP}^3}\, dP$$

in which $f(P)$ is summable in the Lebesgue sense, is

(c) *that the absolute continuity of the integral $F(r_i, s)$ be uniform for all i.*

The condition (c) implies (a). Hence if $u(M)$ satisfies (c), it is given by the formula (III). But

$$F(s) = \lim F(r_i, s)$$

being the limit of a sequence of functions whose absolute continuity is uniform, is itself absolutely continuous. Hence, in this case, the formula (III) reduces to the Poisson integral (II).

Conversely, if u is given by (II), u is the difference of two harmonic functions, u_1, u_2, which are also given by (II) in terms of two non-negative summable functions f_1, f_2, where $f = f_1 - f_2$; u_1 and u_2 are therefore non-negative and

$$\lim u_1(M) = f_1(A); \quad \lim u_2(M) = f_2(A)$$

at nearly every point A, with $\lim (M) = A$.

But the necessary condition that for a non-negative function

$$\int_{\sigma(r_i,s)} u_1(M)\, dM = \int_{\sigma(1,s)} f_1(P)\, dP,$$

is that the absolutely continuity of $\int u_1(M)\, dM$ be uniform.* The same is true of $u_2(M)$ and therefore the absolute continuity of $\int u(M)\, dM$ is uniform.

In precisely the same way as in the corresponding proposition in the case of the circle, the following corollary may be proved.

Corollary. The condition (c) *implies that the absolutely continuity of*

$$\int_{\sigma(r,s)} u(M)\, dM$$

is uniform over all r, $r < 1$.†

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* de la Vallée Poussin, "Sur l'intégrale de Lebesgue," *Transactions of the American Mathematical Society*, Vol. 16 (1915), pp. 435-501. See p. 447.

† G. C. Evans, "Logarithmic Potential. Discontinuous Dirichlet and Neumann Problems," *American Mathematical Society, Colloquium Series*, (in press).

From this theorem we obtain easily the theorem, which is proved in the case of the circle by a different method, by M. Noaillon.*

THEOREM. *A necessary and sufficient condition that $u(M)$ be given by the Poisson integral* (II) *is*

$$(c') \qquad \lim_{i,j=\infty} \int_S |u_i(P) - u_j(P)| \, dP = 0$$

where $u_i(P)$ represents the value of u at the point which is the projection of P from O on the sphere S_i of radius r_i; in other words the sequence of functions $u_i(P)$ is convergent in the mean of order 1.

From the inequality

$$\int_e |u_i(P)| \, dP \leq \int_e |u_j(P)| \, dP + \int_e |u_i(P) - u_j(P)| \, dP, \qquad i > j,$$

it is easily seen that the condition (c') implies (c). For if (c') is true it is possible, given ϵ to take a fixed j so large that the second term of the second member is less than $\epsilon/2$ for any set e on S; and then to choose meas. (e) so small that

$$\int_e |u_i(P)| \, dP < \epsilon/2 \qquad (i = 1, 2, 3, \cdots, j).$$

Thus, for every i, we shall have $\int_e |u_i(P)| \, dP < \epsilon$. Hence the sufficiency of (c') is established.

Conversely, if u is given by (II), $u_i(P)$ approaches $f(P)$ nearly everywhere on S, as we have seen. And the absolute continuity of $\int u_i(P) dP$ is uniform; the same is true of $\int |u_i(P) - f(P)| \, dP$. But

$$\int_S |u_i(P) - u_j(P)| \, dP \leq \int_S |u_i(P) - f(P)| \, dP + \int_S |u_j(P) - f(P)| \, dP;$$

and since the second member of this inequality approaches zero as i and j become infinite, it follows that

$$\lim_{i,j=\infty} \int_S |u_i(P) - u_j(P)| \, dP = 0.$$

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* Noaillon, *Comptes Rendus*, Vol. 182 (1926), p. 1371.

We have now the following result:

THEOREM for the Dirichlet Problem. *Given $f(A)$ summable on S there is one and only one function of the class defined by condition* (c) *or of the class defined by* (c′), *harmonic in Σ, which takes on (in the sense of the theorem of* § 3) *the values $f(A)$ as boundary values almost exerywhere on S. It is given by the Poisson integral* (II).

If $f(A)$ is bounded on S, or bounded except on a point set of measure 0, $u(M)$ is bounded in Σ and vice versa. In this case condition (c) is satisfied. *Hence there is one and only one bounded function, harmonic in Σ which takes on $f(A)$ almost everywhere on S, where $f(A)$ is bounded and measurable in the Lebesgue sense.*

We have already shown that any function which satisfies (c) and is harmonic within S takes on boundary values almost everywhere on S.

The case just considered is a special case of that where $|\, u(M) \,| \leq \psi(A)$ and $\psi(A)$ is summable over S. In fact, under this inequality, the absolute continuity of $\int u(M)\,dM$ is uniform. Another special case of this same condition is that where $|\, \partial u/\partial r \,|$ is summable over Σ, a situation where the $\psi(A)$ is not necessarily a mere constant. For we have

$$\int_{r_0}^{r} dr \int_{\sigma(r,s)} \frac{1}{r^2}\, \frac{\partial u}{\partial r}\, d\sigma = \int_{\sigma(r,s)} u(r, M)\, dM - \int_{\sigma(r_0,s)} u(r_0, M)\, dM,$$

so that

$$\left| \int_{\sigma(r,s)} u(r, M)\, dM \right| \leq \int_{\sigma(r_0,s)} |\, u(r_0, M) \,|\, dM + (1/r_0^2) \int_{\Sigma_s} \left| \frac{\partial u}{\partial r} \right| d\tau.$$

the volume integral being extended over the conical region with vertex O and base $\sigma(1, s)$. But the right-hand member and $\int u(M)\,dM$ define additive functions of point sets on S_r. Hence

$$\int_{\sigma(r,s)} |\, u(r, M) \,|\, dM \leq \int_{\sigma(r_0,s)} |\, u(r_0, M) \,|\, dM + (1/r_0^2) \int_{\Sigma_s} \left| \frac{\partial u}{\partial r} \right| d\tau.$$

Moreover, since the right-hand member is absolutely continuous, and independent of r, it may be written in the form $\int_{\sigma(1,s)} \psi(P)\, dP$, whence, wherever $\psi(P)$ is the derivative of its own integral,

$$|\, u(r, M) \,| \leq (1/r_0^2)\, \psi(P)\,,$$

P being the projection of M from O on S. But since this condition holds almost everywhere, if we define $\psi(P) = +\infty$ where it is not the derivative of its own integral, the function $\psi(P)$ will still be summable and the inequality will hold throughout.

A further special case, of interest on account of its relation to classical treatments of the Dirichlet problem, is that where the integral

$$D(u) = \int_{\Sigma} \left\{ \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial z} \right)^2 \right\} dM,$$

extended over the interior of Σ, exists. In fact the existence of $D(u)$ implies that $(\partial u/\partial r)^2$ and therefore that $| \partial u/\partial r |$ is summable over Σ.

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Notes on Formal Modular Protomorphs.*

By O. C. Hazlett.

1. *Relation to the literature.* In a recent paper,† W. L. G. Williams has proved several theorems relating to formal modular protomorphs of a certain class of binary forms with respect to the general Galois field. He considers the binary form

$$f(a;x) = a_0x_1^q + \binom{q}{1} a_1x_1^{q-1}x_2 + \binom{q}{2} a_2x_1^{q-2}x_2^2 + \cdots + a_qx_2^q$$

where the a's are independent variables. It is well known that one set of algebraic protomorphs of $f(x,y)$ is $C_1, C_2, \cdots, C_q$ where $C_1 = a_0$, $C_2 = a_0a_2 - a_1^2$, and in general

$$C_{2m} = a_0a_{2m} - \binom{2m}{1} a_1a_{2m-1} + \binom{2m}{2} a_2a_{2m-2} \cdots$$

$$+ (-1)^{m-1} \binom{2m}{m-1} a_{m-1}a_{m+1} + \tfrac{1}{2}(-1)^m \binom{2m}{m} a_m^2$$

$$C_{2m+1} = (a\theta - 2a_1)\, C_{2m}$$

where $\quad \theta = a_1\ \partial/\partial a_0 + a_2\ \partial/\partial a_1 + \cdots + a_q\ \partial/\partial a_{q-1}.$

He then proves that the seminvariants C_i $(i = 1, 2, \cdots, q)$ and β form a fundamental system of protomorphs of the binary q-ic form, mod p, p being a prime such that $\binom{q}{j} \not\equiv 0$, mod p $(j = 1, 2, \cdots, q-1)$. Here $\beta \equiv a_1^p - a_0^{p-1}a_1$. Similarly, a fundamental system of protomorphs of a system of binary forms, mod p (with similar restrictions on p) is a fundamental system of algebraic protomorphs of this system together with one additional formal modular seminvariant, namely β formed for any one of the forms of the system. He proves a similar theorem for formal modular protomorphs with respect to any Galois field, $GF[p^n]$, of order p^n, with the same restriction on p.

Unfortunately, however, his proofs are such that they hold only for a

* Read to the American Mathematical Society, Sept. 1926.

† "Fundamental Systems of Formal Modular Protomorphs of Binary Forms," *Transactions of the American Mathematical Society*, Vol. 28 (1926), pp. 183-197.

form or system of forms for which no binomial coefficient is congruent to zero modulo p and hence one wonders if the theorems are true for the general form or system of forms. Although any integer $< p$ is admissible as a value of q by his theorem, yet for values of $p < 20$, a rather large number of values of $q \leqq 20$ are not admissible, as shown by the following table, in which $A =$ admissible values of q, $I =$ inadmissible values of q:

$p = 2$, $I = 2, 4, 5, 6, 8-14, 16-20$, $A = 3, 7, 15$;
$p = 3$, $I = 3, 4, 6, 7, 9-16, 18-20$, $A = 2, 5, 8, 17$;
$p = 5$, $I = 5-8, 10-13, 15-18, 20$, $A = 2-4, 9, 14, 19$;
$p = 7$, $I = 7-12, 14-19$, $A = 2-6, 13, 20$;
$p = 11$, $I = 11-20$, $A = 2-10$;
$p = 13$, $I = 13-20$, $A = 2-12$;
$p = 17$, $I = 17-20$, $A = 2-16$;
$p = 19$, $I = 19, 20$, $A = 2-18$.

Hence, it may be of interest to observe that his theorems quoted above hold for a form or system of forms provided merely that we exclude forms whose order is divisible by p.

2. *Protomorphs for the algebraic case.* Now the theorem about algebraic seminvariants of a binary form $f(a; x)$ of order q asserts that any algebraic seminvariant S is expressible in the form $a_0^{\mu}F(a_0, B_2, B_3, \cdots, B_q)$ where F is a polynomial in its arguments and μ is an integer, positive, negative or zero. Here $B_2, B_3, \cdots, B_q$ is any set of $q-1$ rational integral seminvariants of f which are such that B_i is of weight i and actually involves a_i.* In classical invariant theory it is usually customary † to use as these B's a special set of seminvariants $C_2, C_3, \cdots, C_q$ in which each is of the lowest possible degree. Those of even weights are of the second degree and those of odd weights are of the third degree. Sometimes, however, one uses the coefficients of f in the special form where the coefficient of the next to the highest power of x_1 is zero. If we do not use binomial coefficients (in this paper we shall not use binomial coefficients), then the transformation which carries $f(a; x) \equiv \sum_i a_i x_1^{q-i} x_2^i$ into this special form is $x_1 = x_1' - a_1 x_2'/qa_0$, $x_2 = x_2'$ and the resulting coefficients are $A_0 = a_0$, $A_1 = 0$, $A_i = a_i + \phi_i$ $(i > 1)$ where each ϕ_i is a polynomial in the a_j $(j < 1)$ divided by a power of a_0. Moreover, the coefficients of each ϕ_i are integers divided by a power

* Elliott, ***Algebra of Quantics***, first edition, p. 213.

† *Ibid.*, pp. 214-215.

of q and thus the transformation is admissible for the modular case if q is not congruent to zero in the field. Also, any algebraic seminvariant S of a system of forms $f_i(a^{(i)}; x)$ $(i = 1, 2, \cdots, k)$ is such that $a_0^{(1)\mu_1} a_0^{(2)\mu_2} \cdots a_0^{(k)\mu_k} S$ is a polynomial in seminvariants of each of the forms f_i taken separately and in the leaders $a_0^{(1)} a_1^{(2)} - a_1^{(1)} a_0^{(2)}$, etc. of the Jacobians of one of the forms f_1 and the rest.*

3. *Modular protomorphs as algebraic protomorphs.* This last theorem is of interest in connection with a theorem about the formal modular invariants of a binary form $f(a; x)$. First, however, we need

LEMMA I. *Any formal modular seminvariant of a form or system of forms with respect to the Galois field, $GF[p^n]$, of order p^n is the sum of a finite number of seminvariants which are pseudo-isobaric.*

For it was proved elsewhere † that any seminvariant is annihilated, modulo p, by Ω' and its p'th, p^{2}'th, $\cdots$, p^{n-1}'th symbolic powers where

$$\Omega' = \Omega + (1/[n]!)\Omega^{[n]} + \cdots + (1/[kn]!)\Omega^{[kn]} + \cdots$$

in which $[in] = p^{in}$, Ω is the first Aronhold operator of classical invariant theory and $\Omega^{[in]}$ denotes the $[in]$'th symbolic power of Ω. The converse is also true. Now if Ω operates on any polynomial in the a's, it decreases the weight by 1 and, in general, $\Omega^{[kn]}$ decreases the weight by p^{kn}. Moreover, Ω replaces an isobaric function by an isobaric function and a pseudo-isobaric function by a pseudo-isobaric one. Hence, if Ω' annihilates ΣS_i modulo p, where each S_i is pseudo-isobaric and no two S_i have weights which are congruent, modulo $p^n - 1$, then Ω' must annihilate each S_i, modulo p. Hence each S_i is a seminvariant.

But elsewhere ‡ it was proved that any isobaric formal modular invariant C of a system, S, of binary forms with respect to the Galois field, $GF[p^n]$, of order p^n is congruent, modulo p, to an algebraic invariant of S. In a later paper § it was proved that either C was congruent to a rational algebraic in-

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* *Ibid.*, p. 218.

† Hazlett, "Annihilators of Modular Invariants and Covariants," *Annals of Mathematics*, Series 2, Vol. 23 (1922), p. 210.

‡ Hazlett, "A Symbolic Theory of Formal Modular Invariants," *Transactions of the American Mathematical Society*, Vol. 24 (1922), p. 300. Referred to as S. T.

§ Hazlett, ***Formal Modular Covariants as Algebraic Invariants.*** (As yet unpublished) § 5. Referred to as A. I.

variant of S or that a suitable power of C is congruent to such a covariant. Since * a pseudo-isobaric covariant C of a binary system S is such that a suitable power of C (or C multiplied by a suitable power of L) is an isobaric covariant of an enlarged system, S', then these results apply also to pseudo-isobaric invariants of S. But these theorems, although stated only for invariants under the total linear group on the x's, yet the same proof holds without change for the invariants of S under any linear group with coefficients in $GF[p^n]$ and hence, in particular, for seminvariants. Thus we have

THEOREM I. *If $f(a;x)$ is any binary form of order q and if S is any formal modular seminvariant of f for the Galois field, $GF[p^n]$, of order p^n, then a suitable power of S is congruent in the field to a polynomial in the algebraic seminvariants of $f_0 = f(a;x)$, $f_1 = f(a^{p^n};x)$, $\cdots$, $f_q = f(a^{q^n};x)$ and in the invariants of the a's under the total linear group on the a's divided by a power of L. Similarly for a system of forms.*

This is equivalent to saying that any such seminvariant S, when multiplied by a suitable power of L is congruent to a polynomial in the algebraic seminvariants of the system f_0, f_1, $\cdots$, f_{q+1}. For those formal modular seminvariants which are congruent to rational integral algebraic seminvariants of these $q+2$ forms, a complete system of protomorphs consists of the seminvariants a_0, A_2, A_3, $\cdots$, A_q of f_0 and also the corresponding seminvariants $a_0^{p^n}$, $A_2^{p^n}$, $\cdots$ of the other f_i together with the leaders of certain covariants, viz., the Jacobians $J_1 \equiv a_0 a_1^{p^n} - a_0^{p^n} a_1$, $J_2 \equiv a_0 a_1^{p^{2n}} - a_0^{p^{2n}} a_1$, etc. of f_0 and each of the other forms. But the seminvariants $a_0^{p^n}$, $A_2^{p^n}$, etc. are positive integral powers of a_0, A_2, etc. and it is readily seen that

$$J_2 = (a_0 J_1^{p^n} + a_0^{p^{2n}} J_1) / a_0^{p^n},$$
$$J_3 = (a_0 J_1^{p^{2n}} + a_0^{p^{3n}} J_1) / a_0^{p^{2n}},$$

and, in general, that

$$J_l \equiv (a_0 J_1^{p^{(l-1)n}} + a_0^{p^{ln}} J_{l-1}) / a_0^{p^{(l-1)n}}.$$

Thus every J_l is expressible as a polynomial in J_1 and a_0 divided by a power of a_0. Thus we have the following theorem for those formal modular seminvariants of f which are congruent to rational integral algebraic seminvariants of f_0, f_1, $\cdots$ f_{q+1}.

THEOREM II. *Let $f(a;x)$ be any binary form of order q and consider*

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* S. T., pp. 301-303; A. I., § 5.

its formal modular seminvariants with respect to the Galois field, $GF[p^n]$, *of order* p^n. *If* q *is not divisible by* p, *then a complete set of formal modular protomorphs consists of a complete set of algebraic protomorphs of* f, *together with* $a_0a_1^{p^n} - a_0^{p^n}a_1$.

4. *General case.* To prove this theorem for the general case, we can proceed precisely as in the article by W. L. G. Williams cited above, using the A_i instead of the C_i. Or we can proceed as in the proof of the corresponding theorem for algebraic seminvariants.

In the second of these proofs, we note that since the formula for A_i can be solved for a_i in terms of A_i and the a_j $(j < i)$, we can readily express any formal modular seminvariant, S, as a polynomial, S_1, in the A_i and a_0, a_1 divided by a power of a_0. Now S_1 can involve a_1 only through the function $a_1^{p^n} - a_0^{p^n-1}a_1$. For, since S_1 is identically equal to S, it is annihilated modulo p by Ω' and thus, if

$$S_1 = a_0^{-\mu}F(a_0, a_1, A_2, \cdots, A_q)$$

where F is a polynomial in the arguments, then

$$\Omega' S_1 = a_0^{-\mu}\left[q\delta F + \sum_{ij} C_i a_i \frac{\partial F}{\partial A_j} \frac{\partial A_j}{\partial a_{i+1}} \right]$$

where each C_i is a numerical coefficient, where

$$\delta = a_0 \frac{\partial}{\partial a_1} + a_0^{[n]} \frac{\partial}{a_1^{[n]}} + \cdots + a_0^{[(m+1)n]} \frac{\partial}{\partial a_1^{[(m+1)n]}}$$

and where i, j range over the values $1, 2, 3, \cdots, q$. Hence

$$\Omega' S_1 = q\, a_0^{-\mu}\, \delta F \equiv 0 \qquad (\text{mod } p).$$

In this connection, we need

LEMMA 2. *Let* F *be a polynomial in* a_0, a_1, *and variables* A_2, A_3, *etc. in which* a_0 *and* a_1 *are the first two coefficients of the binary form* f *of order* q, *but in which* A_2, A_3, *etc. may be any variables which may assume any set of values, independent of the values assumed by* a_0 *and* a_1. *Then* F *is annihilated by the differential operator* δ *given above if and only if* F *is expressible as a polynomial in* a_0, *the* A_i *and* $a_1^{p^n} - a_0^{p^n-1}a_1$ *divided by a positive integral power of* a_0.

Let $\qquad t_0 = a_0^{l_0} a_1^{K_0} \phi_0(A_2, \cdots, A_q)$

where $\qquad K_0 = k_0 + k_0' p^n + k_0'' p^{2n} + \cdots$

be the term of F which is of lowest degree in a_0 and let k be a non-negative integer less than p^n. The result of operating on t with $a_0 \dfrac{\partial}{\partial a_1}$ is congruent to

(1) $$k_0\, a_0^{l_0+1}\, a_1^{K_1}\, \phi_0$$

where $$K_1 = (k_0 - 1) + k'_0 p^n + k_0'' p^{2n} + \cdots$$

and, accordingly, since F is annihilated modulo p by δ, the F must contain a term t_1 which is such that, when t_1 is operated on by $a_0^{p^n} \dfrac{\partial}{\partial a_1^{p^{2n}}}$, or by $a_0^{p^{2n}} \dfrac{\partial}{\partial a_1^{p^n}}$ etc. it produces precisely (1) above. Hence (1) would have to come from a term whose degree in a_0 is at most $l_0 - (p^n - 1)$, which is contrary to the definitnon of t_0. Hence $k_0 = 0$.

But the result of operating on t_0 with $a_0^{p^n} \dfrac{\partial}{\partial a_1^{p^n}}$ is

(2) $$k_0'\, a_0^{l_0+p^n}\, a_1^{K_2}\phi_0$$

where

$$K_2 = (k_0' - 1)p^n + k_0'' p^{2n} + \cdots .$$

Since F is annihilated by δ, F must contain a term t_2 which is such that, when t_2 is operated upon by $a_0 \dfrac{\partial}{\partial a_1}$ or $a_0^{p^{2n}} \dfrac{\partial}{\partial a_1^{p^{2n}}}$ or etc., it provides (2). But, since t_0 is the term of lowest degree in a_0, none of these alternatives is possible except the first, and thus

$$t_2 = -\, k_0'\, a_0^{l_0+p^n-1}\, a_1^{K_3}\phi_0$$

where

$$K_3 = 1 + (k_0' - 1)p^n + k_0'' p^{2n} + \cdots$$

$I \neq k_0' \neq 0$, this means that the term of F which is of lowest degree in a_0 is accounted for by

$$t_0 + (1/k_0')t_2 = (a_1^{p^n} - a_0^{p^n-1}\, a_1)\, a_0^{l_0}\, a_1^{K_2}\phi_0;$$

whereas, if $k_0' = 0$, we proceed to consider the result of operating on t_0 with $a_0^{p^{2n}} \dfrac{\partial}{\partial a_1^{p^{2n}}}$

$$\text{Now } \delta[t_0 + (1/k_0')t_2] = [\delta(a_1^{p^n} - a_0^{p^n-1}a_1)]\,[a_0^{l_0}a_1^{K_2}\phi_0] + [a_1^{p^n} - a_0^{p^n-1}a_1)]\,\delta\,[a_0^{l_0}a_1^{K_2}\phi_0]$$

But $\delta(a_1^{p^n} - a_0^{p^n-1}a_1) = 0$ and

$$\delta(a_0^{l_0} a_1^{K_2}\phi_0)$$
$$= [(k_0' - 1)a_0^{l_0+p^n}a_1^{K_2-p^n} + k_0''a_0^{l_0+p^{2n}}a_1^{K_2-p^{2n}}$$
$$+ k_0'''a_0^{l_0+p^{3n}}a^{K_2-p^{3n}} + \cdots]\,\phi_0.$$

Thus δ operating on $t_0 + (1/k_0')t_2$ gives rise only to terms whose degree in a_0 is higher than that of t_0 and not less than that of t_2. Moreover, any such term is of the form

$$t_3 = C\,a_0^{l_1} + a_1^{K_1'}\phi_0$$

where $l_1 = l_0 + p^{in}$ and $K_1' = k_1'p^n + k_1''p^{2n} + \cdots + k_1^{(m)}p^{mn}$ or is of the form $a_0^{p^n-1}a_1t_3$. Here i and each of the k' is a non-negative integer subject to the restrictions that $1 \leqq i \leqq m$ and $k_1^{(j)} \leqq k_0^{(j)}$, and C is a constant. Note that t_3 (or $a_0^{p^n-1}a_1t_3$) arises from $t_0 + 1/k_0'\ t_2$ only by operating with $a_0^{[i]}\ \dfrac{\partial}{\partial a_1^{[i]}}$.

Since F is annihilated by δ, then the totality F_1 of terms of F not accounted for by $t_0 + (1/k_0')t_2$ must be annihilated by δ. Now proceed with F_1 as we did with F. Since the degree of F_1 in a_0 is higher than the degree of F in a_0, then at the end of a finite number of steps we arrive at a pair τ of terms of the same general type as $t_0 + (1/k_0')t_2$ of which the term of the highest degree in a_0 is of degree 1 in a_1. By the foregoing argument, this and the pairs of terms obtained by the preceding argument exhaust all terms of F except that conceivably it may not account for precisely all the coefficient of the term in the highest power of a_0. But if there were a term in the highest power of a_0—say t_5—which is not accounted for by the pair τ, then when we operate on t_5 by δ we should get a term, s_5, independent of a_1. Now the only way that s_5 can be cancelled is by $-s_5$ arising from differentiating a term t_6 of F. By the foregoing argument, t_6 must be of higher degree than t_5 in a_0. Since t_5 is a term of F which is of the lowest degree in a_1, this is impossible and thus the last pair τ precisely exhausts all remaining terms of S. Combining results, we see that the totality of terms of F which contain a_1 as factor is divisible by $a_1^{p^n} - a_0^{p^n-1}a_1$.

Moreover, since

$$(3)\quad \delta(a_1^{p^n} - a_0^{p^n-1}a_1)Q = Q\delta\,(a_1^{p^n} - a_0^{p^n-1}) + (a_1^{p^n} - a_0^{p^n-1}a_1)\delta Q\,,$$
$$= (a_1^{p^n} - a_0^{p^n-1}a_1)\delta Q\,,$$

it follows that the quotient, Q, is annihilated by δ. By induction, F is a polynomial in a_0, A_2, A_3, etc. and $a_1^{p^n} - a_0^{p^n-1}a_1$, and the lemma is proved.

Hence we at once have Theorem II for a single binary form. But since the protomorphs A_i are all expressible as rational functions of the S_j where the S_j form a complete set of algebraic protomorphs of f, we have

COROLLARY I. *Under the hypothesis of the theorem, a complete set of formal modular protomorphs of f is any complete set of algebraic protomorphs and* $a_1^{p^n} - a_0^{p^n-1}a_1$.

Similarly, we have

THEOREM III. *Let $f(a;x)$, $g(b;x)$, $h(c;x)$, $\cdots$ be any system of binary forms and consider the formal modular seminvariants of this system with respect to the Galois field, $GF[p^n]$, of order p^n. If the order of no one of the ground forms is divisible by p, then a complete set of formal modular protomorphs is any complete set of algebraic protomorphs of the system together with* $a_0a_1p^{p^n} - a_0^{p^n}a_1$.

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Irreducible Continuous Curves.*

BY HARRY MERRILL GEHMAN.

A point set M is said to be *an irreducible continuous curve about a point set* A, if M is a continuous curve and contains A, but contains no proper subset which is a continuous curve and contains A. In general, a point set M is said to be *irreducible* with respect to a property X, if M has property X, but contains no proper subset which has property X.

In the present paper, we shall first show that a set which is both a continuous curve and an irreducible continuum about a point set A,† is also an irreducible continuous curve about A. Since every continuum is an irreducible continuum about itself, it follows that if the set A is properly selected, *every* continuous curve is an irreducible continuous curve about a set A. We are therefore principally concerned in this paper with some properties of the set A for a given continuous curve (Theorems 2, 3, and 4), and with the possibility of constructing a continuous curve which is irreducible about a given set A (Theorems 5 and 6). Some rather important results concerning continuous curves are obtained incidentally (Corollaries 4B, 4C, and 4D).

THEOREM 1. *A point set which is both a continuous curve and an irreducible continuum about a point set A is also an irreducible continuous curve about A. Conversely, an irreducible continuous curve about a point set A is an irreducible continuum about A.*

Proof. If M is an irreducible continuum about a set A, then no proper subcontinuum of M contains A, whether the subcontinuum be a continuous curve or not. Therefore if M is also a continuous curve, it is an irreducible continuous curve about A.

Suppose the converse were not true, and there existed a set M which is

* Presented to the American Mathematical Society, Feb. 27, 1926.

† This generalization of the notion of an irreducible continuum between two points is due to W. A. Wilson, "On the oscillation of a continuum at a point," *Transactions of the American Mathematical Society*, Vol. 27 (1925), pp. 429-440. See also H. M. Gehman, "Concerning irreducibly connected sets and irreducible continua," *Proceedings of the National Academy of Sciences*, Vol. 12 (1926), pp. 544-547. We shall refer to this paper hereafter as I. C. S.

an irreducible continuous curve about a set A, but is not an irreducible continuum about A. Then M contains a proper subcontinuum N containing A. Let P be a point of $M - N$, and let C be a circle about P whose interior contains no points of N. If X denotes the points of M in the interior of C, the set K which is the maximal connected subset of $M - X$ that contains N, is a continuous curve.* That is, M contains a proper subset K which is a continuous curve and contains A, which is impossible since we assumed that M was an irreducible continuous curve about A. Therefore, M is an irreducible continuum about A.

THEOREM 2. *Every continuous curve is an irreducible continuous curve about the set consisting of its non-cut-points.*†

Proof. In Theorem 4 of our paper I. C. S., we have proved that every bounded continuum M is an irreducible continuum about the set consisting of its non-cut-points. Therefore if M is a continuous curve, it follows from Theorem 1 that M is an irreducible continuous curve about the set consisting of its non-cut-points.

THEOREM 3. *A necessary and sufficient condition that a continuous curve M be an irreducible continuous curve about one of its subsets A, is that A'* ‡ *contain all the non-cut-points of M.*

Proof. Any continuum containing A will contain A'. Therefore, if A is any subset of M which is such that A' contains all the non-cut-points of M, then the only continuous curve lying in M that can contain A is M itself, by Theorem 2. Therefore the condition is sufficient.

The condition is necessary, for suppose M is an irreducible continuous curve about a set A, but some point P which is a non-cut-point of M, is not contained in A'. We shall show that this leads to a contradiction. There are two cases to be considered, according as P is an end-point of M, or a point of a simple closed curve in M.

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* H. M. Gehman, "Some relations between a continuous curve and its subsets," to appear in the *Annals of Mathematics*. See especially Theorem 6. We shall refer to this paper hereafter as S. R.

† If M is a connected point set, and P is a point of M, then if $M - P$ is not connected, P is said to be a *cut-point* of M; if $M - P$ is connected, P is said to be a *non-cut-point* of M. See R. L. Moore, "Concerning the cut-points of continuous curves, etc.", *Proceedings of the National Academy of Sciences*, Vol. 9 (1923), pp. 101-106.

‡ If A is any point set, A' denotes the set consisting of A and all other points which are limit points of A.

Case 1. Suppose P is an end-point * of M. Let ϵ be a positive number which is less than the distance from P to a point of A'. Then by Menger's definition, there exists a point Q which is such that M can be expressed as the sum of two continuous curves † having only Q in common, one of which contains P but no point of A', and the other contains A' but not P. In that case, a proper subset of M is a continuous curve and contains A, which is contrary to our supposition that M is an irreducible continuous curve about A.

Case 2. Suppose P is a point of a simple closed curve J of M. Let C_1 be a circle about P as center, excluding all points of A', and let C_2 be a circle about P as center whose radius is less than that of C_1. Let QR be an arc of J lying entirely within C_2, and let N be the maximal connected subset of M in C_2 plus its interior, that contains QR. As we have shown before, N is a continuous curve, and therefore $M - N$ contains only a finite number of maximal connected subsets containing points of A, because only a finite number of these subsets can be of diameter greater than the difference in radii of C_1 and C_2.* Let these be $D_1, \cdots, D_k$. By Theorem 7 of our paper S. R., the set D_i' is a continuous curve. Each of these continuous curves has a point in common with N, and every such common point is a point of C_2. Let P_i be a point common to D_i' and N, and for each point P_i select a definite arc P_iQ_i in N, such that Q_i is the only point that the arc has in common with QR. The set $D_1' + D_2' + \cdots + D_k' + P_1Q_1 + P_2Q_2 + \cdots + P_kQ_k + QR$ is a continuous curve containing A, and must therefore be identical with M. In that case, the points of M in the interior of C_2 lie on a finite set of arcs, and we can select two points X and Y of QR such that there are no points of $Q_1 + Q_2 + \cdots + Q_k$ between them. The set $(M - XY)'$ is a continuous curve (by Theorem 7 of S. R.) which contains A and is a proper

* We are using *end-point* here in the sense defined by R. L. Wilder, "Concerning continuous curves," *Fundamenta Mathematicae*, Vol. 7 (1925), pp. 340-377. For the equivalence of Wilder's definition with a modification of one due to Menger, see Theorem 2 of our forthcoming paper entitled, "Concerning end-points of continuous curves and other continua." The modification of Menger's definition may be stated thus: A point P of a continuous curve M is said to be an end-point of M, if given any positive number ϵ, there exists a simple closed curve of diameter less than ϵ, enclosing P, and having only one point in common with M. See Karl Menger, "Grundzüge einer Theorie der Kurven," *Mathematische Annalen*, Vol. 95 (1925), pp. 277-306. The theorem that an end-point of a continuous curve M is a non-cut-point of M which belongs to no simple closed curve in M is due to W. L. Ayres.

† This follows from Theorem 6 of our paper S. R.

‡ W. L. Ayres, "Concerning the arcs and domains of a continuous curve," (abstract), *Bulletin of the American Mathematical Society*, Vol. 32 (1926), p. 37.

subset of M. But this is contrary to our supposition concerning M. Therefore the condition is necessary.

Definition. If a set M is an irreducible continuous curve about a closed set A, but not about any proper closed subset of A, then A is said to be a *basic closed set* about which M is an irreducible continuous curve.

From this definition and Theorem 3, we have the following corollary.

COROLLARY 3A. The only basic closed set about which a continuous curve is an irreducible continuous curve, is the set consisting of all the non-cut-points of the continuous curve and all points which are limit points of non-cut-points.

COROLLARY 3B. If a set M is an irreducible continuous curve about a closed set A, then M is irreducibly connected about A.

Proof. By Theorem 3, if M is an irreducible continuous curve about a closed set A, then A contains all the non-cut-points of M. By Theorem 2 of our paper I. C. S., the set M is irreducibly connected about any set that contains all the non-cut-points of M, and therefore M is irreducibly connected about A.

THEOREM 4. *If a set M is an irreducible continuous curve about a set A, then every subcontinuum K of M is connected im kleinen at every point not in A', and if K contains every point of a maximal connected subset B of A', then K is also connected im kleinen at every point of B.*

Proof. Suppose K were not connected *im kleinen* at a point P of $M - A'$. Then we can construct a circle C about P excluding all points of A'. Within C there exists the state of affairs described in R. L. Moore's characterization of continua which are not continuous curves,* and by the argument given in the proof of Theorem 1 of our paper S. R., it follows that M contains two arcs $P'P''$, $Q'Q''$ which have no points in common, are interior to C, and are such that each of them has points in common with each of the sets M_i for i equal to or greater than some fixed number n, each of the sets M_i being an arc. Therefore the set $P'P'' + Q'Q'' + M_n + M_{n+1}$ contains a simple closed curve J which is interior to C. Every point of a simple closed curve in M is either a non-cut-point of M, or a limit point of non-cut-points, †

* R. L. Moore, "Report on continuous curves from the viewpoint of analysis situs," *Bulletin of the American Mathematical Society*, Vol. 29 (1923), pp. 289-302.

† R. L. Moore, "Concerning the cut-points of continuous curves, etc.", *loc. cit.*, Theorem B*.

and by Theorem 3, all such points must be points of A'. Therefore A' contains J. But this is impossible, as all points of A' are exterior to C, and J is interior to C. Therefore K is connected *im kleinen* at every point of $M - A'$.

Suppose K contains every point of a maximal connected subset B of A', but is not connected *im kleinen* at a point P of B. Then there exists in K the state of affairs described in the preceding paragraph, where the continuum of condensation W (or $\bar{M}$, in Moore's notation) at points of which K is not connected *im kleinen*, contains P. Since we have shown that K is connected *im kleinen* at every point not in A', it follows that W is a subset of A'. Since W contains P, the continuum W is a subset of B.

If on the arc $P'P''$ * we select any point X different from P'', there is a point P_i of one of the sets $M_i (i > n)$ on the arc XP''. The set $P'P_i + M_n + M_i + Q'Q''$ contains a simple closed curve containing the arc $P'P_i$. As we have shown above, such an arc lies in A', and since X was an arbitrary point of $P'P''$, it follows that the entire arc $P'P''$ lies in A'. Since the arc $P'P''$ has the point P'' in common with W, the arc $P'P''$ is also a subset of B and therefore of K. But this contradicts part (2) of Moore's characterization. Therefore K is connected *im kleinen* at every point of B.

COROLLARY 4A. If M is an irreducible continuous curve about a set A, every maximal connected subset of A' is a continuous curve, and not more than a finite number of maximal connected subsets of A' are of diameter greater than any given positive number.

Proof. That every maximal connected subset of A' is a continuous curve, follows directly from Theorem 4.

Suppose that an infinite number of these continuous curves were of diameter greater than some positive number ϵ. Then let us construct in each of these, an arc of diameter greater than ϵ, thus obtaining an infinite collection of arcs, such that no arc in A' joins points of any two of them. From this collection let us select a sequence of arcs approaching a limiting set W.† Since M is connected *im kleinen* at every point of W, there exists in M an arc $P'P''$ which has only P'' in common with W, but which has points in common with infinitely many of the arcs of the given sequence. Then, as was pointed out in the proof of Theorem 4, the arc $P'P''$ belongs to A'. But

* The point P'' is the only point which this arc has in common with W.

† Compare pp. 40-41 of H. M. Gehman, "Concerning the subsets of a plane continuous curve," *Annals of Mathematics*, Vol. 27 (1925), pp. 29-46.

this contradicts our supposition that the arcs were constructed so that no arc in A' joins points of any two of them.

COROLLARY 4B. If H denotes the set of all non-cut-points of a continuous curve, then every maximal connected subset of H' is a continuous curve, and not more than a finite number of maximal connected subsets of H' are of diameter greater than any given positive number.

Proof. The continuous curve is irreducible about H' by Theorem 3, and then the conclusion follows by Corollary 4a.

The following example shows that if K denotes a maximal connected subset of the non-cut-points of a continuous curve M, it is not necessarily true that K' is a continuous curve.

Let M_1 denote the lines $y = 0$ and $y = 1/2^n$ $(n = 0, 1, 2, \cdots)$ between $x = 0$ and $x = 1$. Let M_2 denote the lines $x = k/2^n$ $(n = 0, 1, 2, \cdots$, and $k = 0, 1, \cdots, 2^n)$ between $y = 0$ and $y = 1/2^n$. Let M_3 denote the lines $y = 5/2^{n+3}$ and $y = 7/2^{n+3}$ $(n = 0, 1, 2, \cdots)$ between $x = k/2^n$ and $x = (k/2^n) + (1/2^{n+3})$, for $k = 1, 2, \cdots, 2^n$. The set $M = M_1 + M_2 + M_3$ is a continuous curve, and every point of $M - M_3$ is a non-cut-point of M, while every point of M_3 is a cut-point of M except the points whose x-coordinates are $(k/2^n) + (1/2^{n+3})$ for the values of n and k given above. If K is the maximal connected subset of the non-cut-points of M, that contains the point $(1, 1)$, the set K' evidently contains the points $(1, 0)$ and $(1, 1/2^n)$, for $n = 0, 1, 2, \cdots$. But no two of these points can be joined by an arc in K' unless the arc has points in common with the line $x = 0$. Therefore K' is not connected *im kleinen* at the point $(1, 0)$, and is therefore not a continuous curve. On the other hand, the maximal connected subset of H' that contains the point $(1, 1)$, is the set $M_1 + M_2$ which is a continuous curve.

COROLLARY 4C. If M is an irreducible continuous curve about a set A, and if every subcontinuum of A' is a continuous curve, then every subcontinuum of M is a continuous curve.

Proof. If K is any subcontinuum of M, then K is connected *im kleinen* at every point of $M - A'$, by Theorem 4. If K is not connected *im kleinen* at a point of A', there exists in K the state of affairs described in the proof of Theorem 4. As we pointed out there, each of the arcs M_i (for i greater than n) contains a subarc A_iB_i which has only A_i in common with $P'P''$ and only B_i in common with $Q'Q''$. Each of these arcs lies in a simple closed

curve in M, and is therefore contained in A'. Since $P'P''$ is also in A', the infinite collection of arcs A_iB_i lies in one maximal connected subset B of A'. But in that case, some subcontinuum of B is not a continuous curve,* which is contrary to hypothesis.

COROLLARY 4D. A necessary and sufficient condition that every subcontinuum of a continuous curve M be a continuous curve, is that every subcontinuum of H' be a continuous curve, where H denotes the set of all non-cut-points of M.

Proof. The necessity of the condition is obvious. The sufficiency of the condition follows from Theorem 3 and Corollary 4c.

THEOREM 5. *If A is a subset of a continuous curve S, then a necessary and sufficient condition that S contain an irreducible continuous curve about A, is that every maximal connected subset of A' be a continuous curve and that not more than a finite number of the maximal connected subsets of A' be of diameter greater than any given positive number.*

Proof. The condition is necessary, for if S contains a continuous curve M which is irreducible about a set A, then A' satisfies the given condition by Corollary 4a.

To show that the condition is sufficient, we shall indicate how to construct in S a continuous curve M which is irreducible about a given set A satisfying the condition. In our paper, *Concerning acyclic continuous curves,*† we give a method (in the proof of Theorem 4) for constructing an acyclic continuous curve about a certain subset of a continuous curve. This method bears a close relation to the proof of Theorem 1 of our paper, *On extending a continuous* $(1-1)$ *correspondence of two plane continuous curves to a correspondence of their planes.*‡ If we modify this method so that it bears a similar relation to Theorem 2 of the paper just mentioned, we obtain a method for constructing in S a continuous curve M containing A. Since the method is such that every point of $M - A'$ is a cut-point of M, it follows from Theorem 3 that the continuous curve M is irreducible about A.

THEOREM 6. *A necessary and sufficient condition that a bounded con-*

* H. M. Gehman, "Concerning the subsets of a plane continuous curve," *loc. cit.*, Theorem V.

† To appear in the *Transactions of the American Mathematical Society.* An acyclic continuous curve is one which contains no simple closed curve.

‡ *Transactions of the American Mathematical Society*, Vol. 28 (1926), pp. 252-265.

tinuum S contain an irreducible continuous curve about every subset of S is that every continuum of S be a continuous curve.

Proof. The condition is sufficient, because if A is any subset of S, then S contains a set M which is an irreducible continuum about A.* If every subcontinuum of S is a continuous curve, then M is a continuous curve, and therefore is an irreducible continuous curve about A, by Theorem 1.

Also, the condition is sufficient, because if every subcontinuum of S is a continuous curve, every subset A of S will be such that A' satisfies the conditions of Theorem 5,† and therefore S will contain an irreducible continuous curve about A.

The necessity of the condition will follow as a corollary of Theorem 7.

THEOREM 7. *If a bounded continuum S contains an irreducible continuous curve about every closed disconnected subset A', then every subcontinuum of S is a continuous curve.*

Proof. Let K be any proper subcontinuum of S, and let P be a point of $M - K$. The set $A' = K + P$ is closed and disconnected, and therefore S contains an irreducible continuous curve about A', by hypothesis. Since the set K is a maximal connected subset of A', it is a continuous curve, by Corollary 4a. Therefore every proper subcontinuum of S is a continuous curve. It remains to be shown that S itself is a continuous curve.

If S is an irreducible continuum about a set A' consisting of two points P and Q, then by Theorem 1, S itself is the only subcontinuum of S which can be an irreducible continuous curve about the set A'. Therefore S is a continuous curve.

If S is not an irreducible continuum about any set A' consisting of two points P and Q, then S is decomposable, and by Theorem 2 of our paper S. R., the continuum S is a continuous curve, since we have proved above that every proper subcontinuum of S is a continuous curve.

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* W. A. Wilson, *loc. cit.*, p. 433.

† H. M. Gehman, "Concerning the subsets of a plane continuous curve," *loc. cit.*, Theorem V.

The Plane Quintic With Five Cusps.

By Marguerite Lehr.

The quintic with five cusps (characterized by the Plücker numbers $m = n = \kappa = i = 5$; $\nu = \tau = 0$; $p = 1$) has been considered by del Pezzo,* Field † and Basset.‡ Field gives a general descriptive account of the appearance of the curve under various conditions on the coefficients in its equation; Basset mentions it as the limiting case under quintics with five nodes; but neither paper gives a detailed study of the curve. In such a study the del Pezzo work is fundamental. Starting with the fact that a quintic with five cusps may be obtained by quadratic transformation from a quartic with two cusps, in- and circumscribed to the triangle of reference, he proves that the quintic is uniquely determined by its five cusps. Considering the two cusps of the quartic as the two intersections of the line $\sum\limits_{xyz} x = 0$ with the conic $\sum\limits_{\substack{xyz\\abc}} ayz = 0$ circumscribed to the triangle of reference ABC, he shows that the quartic is uniquely determined by the conditions for contact with $x = 0$, $y = 0$, $z = 0$, respectively, and passage through A, B and C. By means of a quadratic transformation, he obtains the equation of the quintic with cusps at $A\,B\,C$ and at K_1K_2, the intersections of the line $\Sigma\, \alpha x = 0$ with the conic $\Sigma\, yz = 0$, in the form: §

$$(1) \qquad 16xyz(\Sigma\, \alpha x)^2 - 4(\Sigma\, \alpha x)(\Sigma\, yz)(\Sigma\, ayz) - (\Sigma\, yz)^2 \Sigma\, (b-c)^2 x = 0$$

where $\alpha = -a + b + c$; $\beta = a - b + c$; $\gamma = a + b - c$. All the forms involved are symmetric in x, y, z and a, b, c. Write

* P. del Pezzo, "Equazione d'una curva del quinto ordine dotata di cinque cuspidi," *Rendiconti dell' Academia delle Scienze Fisiche e Matematiche* (*di Napoli*), Serie 2, Volume 3, (1889), pp. 46-49.

† Peter Field, "On the Form of a Plane Quintic with Five Cusps," *Transactions of the American Mathematical Society*, Volume 7, (1906), pp. 26-32.

‡ A. B. Basset, "On Quinquenodal and Sexnodal Quintics," *Quarterly Journal of Mathematics*, Volume 37, pp. 199-214; "On Quintic Curves with Four Cusps," *Rendiconti del Circolo Matematico di Palermo*, Volume 26 (1908), pp. 332-335.

§ Slight inaccuracies in some of the forms in the del Pezzo article have been corrected here.

$$\sum (b-c)^2 = l$$
$$\sum ax = k$$
$$\sum yz = \phi$$
$$\sum ayz = \psi.$$

The equation then assumes the form:

$$16xyzk^2 - 4k\phi\psi - \phi^2 l = 0. \tag{2}$$

Del Pezzo gives also the cuspidal tangents at A, B, C, the typical form being the equation of the cuspidal tangent at A:

$$y(\gamma + a) = z(\beta + a). \tag{3}$$

This completes the account of results obtained in his paper.

In the present investigation, which takes as starting point the del Pezzo work, it is proved that if five points in the plane are given as cusps on a quintic, the curve thus uniquely determined is unipartite, with the five cusps and five inflexions occurring alternately. Moreover, it is shown that from the five given points the cuspidal tangents, the inflexions, the inflexional tangents, as well as a series of ordinary points on the quintic, may be obtained by linear constructions.

The method of treatment is based on a slight modification of the form of (2). Replace ψ and yz by:

$$\psi \equiv ayz + \beta zx + \gamma xy = a\phi - 2xh$$

where $h \equiv (c-a)y + (b-a)z$

$$yz = \phi - x(y+z) \equiv \phi - xg.$$

Then the equation for the quintic Q may be written:

$$\phi^2(l + 4ak) - 8xk\phi(h + 2k) + 16x^2k^2 g = 0. \tag{4}$$

Here the importance of the system of conics depending on ϕ and xk is suggested at once. Consequently, setting apart the cusp A, consider the pencil of conics through $B\,C\,K_1K_2$.

The Parametric Treatment of the Quintic. Five given points are to be made cusps on a quintic, which is then uniquely determined. Take as $B\,A\,C$, triangle of reference, any set of three cusps adjacent in that order on the conic ϕ determined by the five points.* The two remaining points determine

*** For the analytical work, it is not necessary that BAC be adjacent on ϕ; for the interpretation, however, such a choice is convenient since it gives a, b, c all positive.**

the line $k \equiv \Sigma\, ax = 0$. No restriction is imposed on the curve either by such a choice of $A\,B\,C$, or by writing ϕ as $\Sigma\, yz = 0$.

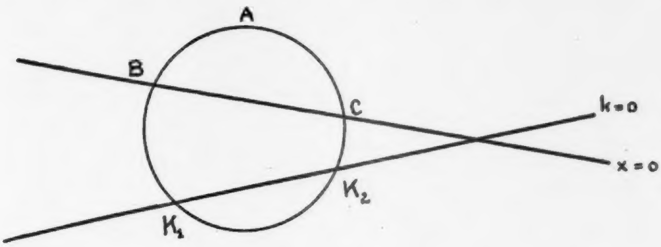

FIG. 1.

The equation of the quintic is then equation (4), where a, b, c are all positive. The cusps $K_1\,K_2$ are given by the lines AK_1, AK_2:

$$by^2 + ayz + cz^2 = 0.$$

The reality of K_1K_2 depends on $a^2 - 4bc$; call this σ^2.

Any conic of the pencil $S_2(\lambda)$

$$\lambda\phi - 4xk = 0 \tag{5}$$

has ten intersections with the quintic, of which eight are fixed at the cusps $B\,C\,K_1K_2$; the two remaining variable intersections are cut out on the conics by the singly-infinite quadratic system of lines $S_1(\lambda^2)$:

$$(l + 4ak) - 2\lambda(h + 2k) + \lambda^2 g = 0 \tag{6}$$

obtained by direct substitution of (5) in (4). To each λ-conic with a given λ-value there corresponds a λ-line, which intersects the conic in two points on Q. The lines (6) envelope a conic Θ:

$$(h + 2k)^2 - g(l + 4ak) = 0$$

which reduces to

$$(b - c)^2\phi - 4axk = 0 \tag{7}$$

or

$$\delta^2\phi - 4axk = 0 \qquad \text{where } \delta = b - c,$$

and is therefore a λ-conic,* corresponding to the value $\lambda = (b - c)^2/a$. The λ-line corresponding to this value is:

*** If $B\,C\,K_1K_2$ are ordinary nodes, the conics through them have eight fixed points of intersection with the Q in question and two variable intersections given by a system of lines; this point of view I hope to develop at some other time. The specialization in this case is the passage of the envelope of the lines through $B\,C\,K_1K_2$, which are therefore cusps.**

(8a) $$l + 4ak - 2(\delta^2/a)(h + 2k) + (\delta^4/a^2)g = 0$$
or
(8b) $$x(4a\alpha - 3\delta^2) + y(\alpha + b - \delta^2/a)^2 + z(\alpha + c - \delta^2/a)^2 = 0.$$

The two points which this line j cuts out on Θ are points on Q; j is tangent to Θ, being a λ-line, and is therefore tangent to Q.

For all points on Θ

$$\frac{l + 4ak}{h + 2k} = \frac{h + 2k}{g} = \rho.$$

The points of contact of λ-lines (6) are given by the value $\rho = \lambda$. In particular, for the point of contact of the line j

(9) $$\frac{l + 4ak}{h + 2k} = \frac{h + 2k}{g} = \frac{\delta^2}{a} \quad .$$

The line j is a special line in the system $S_1(\lambda^2)$; it may be expected to have a special relation to Q. In (4) write

$$l + 4ak = 2(\delta^2/a)(h + 2k) - (\delta^4/a^2)g.$$

The resulting equation

$$[(\delta^2/a)\phi - 4xk]\,\{2\phi(h + 2k) - g[(\delta^2/a)\phi + 4xk]\} = 0$$

gives intersections of Q with the line j. The quadratic factor is accounted for—the line j is tangent to Q on Θ. The cubic cuts out the three remaining points on the line. This cubic meets Θ, i. e. $(\delta^2/a)\phi - 4xk = 0$ on ϕ (at $B\,C\,K_1K_2$) and on the line

$$a(h + 2k) - \delta^2 g = 0\ ;$$

that is, the cubic passes through the point on Θ given by:

$$\frac{h + 2k}{g} = \frac{\delta^2}{a} \quad .$$

This point, however, is the point of contact of j with Θ. The cubic, therefore, cuts the line j at its point of contact with Θ; the line j has three intersections with Q at that point. All singularities are accounted for at the cusps; therefore the line j is an inflexional tangent of Q, and its point of contact with Θ is an inflexion on Q. The cusp A, then, is associated with one of the inflexions by the system of conics through the remaining four cusps. The equation of j, more properly called j_A, is:

(10a) $$a^2x(4a\alpha - 3\delta^2) + y(\sigma^2 + ac)^2 + z(\sigma^2 + ab)^2 = 0.$$

The corresponding inflexion I_A has coordinates

$$x : y : z = \frac{\delta}{a} : -\frac{\sigma^2 + a^2 + 2ac}{\sigma^2 + ac} : \frac{\sigma^2 + a^2 + 2ab}{\sigma^2 + ab}.$$

In the same way, each cusp is connected with an inflexion by the system of conics through the remaining four cusps; e. g. j_B is given by:

$$x(\sigma^2 + bc)^2 + b^2 y\,[4b\beta - 3(c-a)^2] + z(\sigma^2 + ab)^2 = 0 \tag{10b}$$

with the corresponding inflexion I_B

$$x : y : z = \frac{\sigma^2 + b^2 + 2bc}{\sigma^2 + bc} : \frac{c-a}{b} : -\frac{\sigma^2 + b^2 + 2ab}{\sigma^2 + ab}$$

The equation of j_{K_1} is most easily obtained by transforming the triangle of reference to $A\,K_1K_2$, so that the conic $\Sigma\, yz = 0$ becomes the conic $\Sigma\, y'z' = 0$. The points K_1K_2 are so named that K_1 is adjacent to B on ϕ; the coordinates are

$$\begin{aligned} K_1 &: \; x : y : z = -2bc : c(\gamma + \sigma) : b(\beta - \sigma) \\ K_2 &: \; x : y : z = -2bc : c(\gamma - \sigma) : b(\beta + \sigma). \end{aligned}$$

The equations of transformation are:

$$\begin{aligned} \sigma x' &= ax + by + cz && \text{or } k \\ 2ay' &= (\gamma + \sigma)y + (\sigma - \beta)z && \text{or } AK_2 \\ 2az' &= (-\gamma + \sigma)y + (\beta + \sigma)z && \text{or } AK_1 \end{aligned} \tag{11}$$

The new $a'\ b'\ c'$ are the coefficients in the equation of BC referred to the new triangle. This is:

$$-4\sigma^2 x' + (\gamma - \sigma)(\beta + \sigma)y' + (\gamma + \sigma)(\beta - \sigma)z' = 0.$$

So the required values are

$$\begin{aligned} a' &= -4\sigma^2 & \alpha' &= -4a\alpha \\ b' &= (\beta + \sigma)(\gamma - \sigma) & \beta' &= -4\sigma(\sigma + \delta) \\ c' &= (\beta - \sigma)(\gamma + \sigma) & \gamma' &= 4\sigma(-\sigma + \delta). \end{aligned} \tag{12}$$

The equation of j_{K_1} is now given by (10b), with $x'\ y'\ z'$ and $a'\ b'\ c'$ written for $x\ y\ z$ and $a\ b\ c$ respectively,

$$x'(\sigma'^2 + b'c')^2 + b'^2 y'[4b'\beta' - 3(c' - a')^2] + z'(\sigma'^2 + a'b')^2 = 0$$

with equations (11) and (12) as equations of the transformation. The corresponding inflexion I_{K_2} is obtained from I_B by a similar substitution. Then j_{K_2} and I_{K_2} are obtained from j_{K_1} and I_{K_1} by changing the sign of σ.

4

The five inflexions and their inflexional tangents have been found by means of the systems of conics through four cusps. The cuspidal tangents may be expressed in terms of the same systems. If A_1 denote the conjugate of A with respect to $S_2(\lambda)$, i. e. the common point of the polars of A with respect to the conics of $S_2(\lambda)$, then t_A, the cuspidal tangent at A, is the polar of A_1 with respect to ϕ: for:—The polar of A with respect to ϕ is g; the polar with respect to xk is $ax + k$; the common point A_1 has coordinates

$$x : y : z = \delta : -2a : 2a$$

The polar of A_1 with respect to ϕ is:

$$y(2a + \delta) + z(-2a + \delta) = 0\,;$$

that is:
$$y(\gamma + a) - z(\beta + a) = 0$$
which is t_A.

Application of this method gives the equation of the cuspidal tangents at K_1 and K_2. The conjugate of K_1 with respect to the related system of conics through $A\,B\,C\,K_2$ is the common point of its polars with respect to any two conics of the system;

e. g.
$$\text{i)}\quad z[2ax + (\gamma + \sigma)y] = 0$$
$$\text{ii)}\quad y[2ax + (\beta + \sigma)z] = 0.$$

The polar of K_1 with respect to i) is:

$$ax \cdot 2b(\beta - \sigma) + by(\beta - \sigma)(\gamma + \sigma) - cz \cdot 2\sigma(\gamma + \sigma) = 0.$$

The polar with respect to ii) is:

$$ax \cdot 2c(\gamma + \sigma) - by \cdot 2\sigma(\beta - \sigma) + cz(\gamma + \sigma)(\beta - \sigma) = 0.$$

The common point has coordinates

$$x : y : z = \frac{(\gamma + \sigma)(\beta - \sigma) + 4\sigma^2}{-2a} : \frac{b(\beta - \sigma) + \sigma(a - \sigma)}{b} : \frac{c(\gamma + \sigma) - \sigma(a + \sigma)}{c}$$

The polar of this with respect to ϕ is:

$$2abc \sum ax - \sigma \sum a\alpha(b - c)x - \sigma^2 \sum a(b + c)x = 0.$$

This is the cuspidal tangent to Q at K_1. The cuspidal tangent at K_2, obtained by changing the sign of σ, is:

$$2abc \sum ax + \sigma \sum a\alpha(b - c)x - \sigma^2 \sum a(b + c)x = 0.$$

The results obtained may be restated:

The quintic is traced out by pairs of points obtained as the intersections of the conics of the singly-infinite linear system $S_2(\lambda)$ with the corresponding lines of the singly-infinite quadratic system $S_1(\lambda^2)$, all such lines being tangent to the conic Θ of the system $S_2(\lambda)$. For each value of λ there are determined:

1) a λ-conic of the pencil $\lambda\phi - 4axk = 0$;
2) a λ-line tangent to Θ;
3) therefore, two points (common to 1 and 2) on Q;
4) a λ-point (of contact of 2) on Θ.

Each point on Q has its λ-value and has a correspondent with respect to A, i. e. the second point with that same λ-value. Similarly it has a μ-value and a correspondent with respect to B, i. e. determined by means of the conics $\mu\phi - 4yk = 0$ through ACK_1K_2. For a given point $x'y'z'$,

$$\lambda = \frac{4x'k'}{\phi}\,; \qquad \mu = \frac{4y'k'}{\phi}$$

therefore

$$\lambda : \mu = x' : y'$$

A similar correspondence is set up for each of the five cusps. All investigations will be made with respect to the cusp A and then extended to $B\,C\,K_1K_2$.

The curve may be expressed parametrically by means of direct solution of $S_2(\lambda)$ with $S_1(\lambda^2)$. The lines through C thus obtained are given by the equation:

$$(13) \quad \begin{aligned} &x^2[-\lambda(\beta + a)^2 - 4a(a - c)^2 + 4c\delta^2] + y^2[-\lambda(\lambda - a - b)^2] \\ &\quad + 2xy[\lambda^2(\beta + a) - \lambda(2aa - 3a\delta + b\delta) - 2\delta(a^2 - bc)] = 0. \end{aligned}$$

These are pairs of lines joining C to a varying pair of corresponding points P_1P_2, common to a λ-conic and its λ-line. Important properties of the curve may be obtained from this system of lines without proceeding to the complete parametric expression. For the value $\lambda = \delta^2/a$, i. e. at the inflexion I_A, the two lines coincide, for the λ-line given by this value is tangent to its conic; thus I_A is its own correspondent. Similarly, for $\lambda = \infty$, the λ-line $g = 0$ is tangent to its conic ϕ, and the cusp A is its own correspondent. As λ changes continuously from ∞, the two lines CP_1, CP_2, move from coincidence at A, the point P_1 tracing out one branch from the cusp, the point P_2 tracing out the other branch. If this circuit is to be closed, the lines must again come to coincidence; if another circuit exists, another position of co-

incidence of P_1P_2 is necessary as a transition between imaginary and real values. In other words, the number of values of λ which give self-corresponding points determines how many circuits the curve has.

Such values of λ are given by the discriminant of the quadratic (13), set equal to zero. This discriminant is:

$$\Delta \equiv [\lambda^2(\beta + a) - \lambda(2a\alpha - 3a\delta + b\delta) - 2\delta(a^2 - bc)]^2$$
$$+ [-\lambda(\beta + a)^2 - 4a(\alpha - c)^2 + 4c\delta^2]\,[\lambda(\lambda - \alpha - b)^2].$$

Two known zeroes of this expression are $\lambda = (b - c)^2/a$, $\lambda = \infty$; the reduction is therefore simplified. The reduced form of Δ is:

(14) $\Delta = 4(\delta^2 - a\lambda)\,[\lambda^2(a^2 - 3bc) - \lambda\alpha(2a^2 - 5bc) + (a^2 - bc)^2].$

The quadratic factor set equal to zero gives other λ-values which determine self-corresponding points on Q. The reality of such values depends on the discriminant of the quadratic in question, i. e. on

$$a^2(2a^2 - 5bc)^2 - 4(a^2 - 3bc)(a^2 - bc)^2$$

which reduces to

(15) $$-3b^2c^2(a^2 - 4bc).$$

Consideration of the relative values of α, a, b, c will determine the sign of this discriminant. With $ABCK_1K_2$ placed as described, a, b, c are all positive. The points K_1K_2 are given as the intersections of k wth ϕ, i. e. by the lines AK_1, AK_2:

(16) $$by^2 + ayz + cz^2 = 0.$$

If K_1K_2 are to be real,*

(17) $$a^2 > 4bc.$$

Application of (17) to the discriminant (15) shows that the result is negative; the two λ-values given by the quadratic equation are imaginary. Hence there are only two values for λ giving self-corresponding points; there is therefore one circuit only.

The solution of equation (13) may be written in the form:

(18) $Px = (T \pm R)y$

where $P = -\lambda(\beta + a)^2 - 4a(\alpha - c)^2 + 4c\delta^2$

$T = -\lambda^2(\beta + a) + \lambda(2a\alpha - 3a\delta + b\delta) + 2\delta(a^2 - bc)$

$\Delta^{1/2} \equiv R = +\{4(\delta^2 - a\lambda)\,[\lambda^2(a^2 - 3bc) - a\lambda(2a^2 - 5bc) + (a^2 - bc)^2]\}^{1/2}.$

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* The present considerations are limited, in the interests of directness of development, to the case of five real cusps.

This with the value for $x:z$ obtained from $S_1(\lambda^2)$ gives a complete parametric representation of the quintic.

$$(19) \qquad x:y:z=(\lambda-a-c)^2(T\pm R):P(\lambda-a-c)^2$$
$$:(T\pm R)(4a\lambda-4aa-\delta^2)-P(\lambda-a-b)^2.$$

The general investigation of properties of the curve derived from equation (18) is greatly facilitated if the λ-values of special elements of interest are found first by direct use of $S_2(\lambda)$, $S_1(\lambda^2)$.

Conic ϕ: corresponds to the value $\lambda=\infty$; its λ-line is $g=0$, which cuts out A, self-corresponding. The λ-point on Θ, i. e. the contact point of g with Θ is A_1, the conjugate point of A with respect to the pencil $S_2(\lambda)$; for A_1 is

$$x:y:z=\delta:-2a:2a$$

which lies on Θ and on g; but g is a λ-line and therefore tangent to Θ. Its point of contact is then A_1. So Θ is determined as the conic of the pencil $S_2(\lambda)$ through A_1.

Conic Θ: corresponds to $\lambda=(b-c)^2/a$; its λ-line is j_A (the tangent at the inflexion I_A); the λ-point of contact on Θ is I_A, self-corresponding on Q.

Conic $xk=0$: corresponds to $\lambda=0$; its λ-line is $l+4ak=0$, which cuts out on x and on k respectively a point of Q, the fifth point on each, since both x and k have already four points of intersection with Q, viz., x two each at B and C, k two each at K_1 and K_2. These are the conics of importance; the other line pairs in the pencil are given by $\lambda=2(a-\sigma)$ and $\lambda=2(a+\sigma)$, but as only one line pair is needed, xk is selected for simplicity.

Naturally the distribution of the five cusps and five inflexions on the single circuit invites discussion. For this purpose, the λ-values which give BCK_1K_2 are necessary. For a given point on Q, in general, direct substitution in $S_2(\lambda)$ determines the proper λ-value and therefore fixes the λ-line. For the points BCK_1K_2 which are fixed for the pencil $S_2(\lambda)$, however, this process fails to determine λ; these are points of special interest on Q. Here the system $S_1(\lambda^2)$ may be used to advantage, for though in general two λ-lines pass through a point, these points, being on Θ, have only one λ-line passing through them, i. e. the tangent to Θ at the particular point in question. For B, then,

$$x(-4a\lambda+4aa+\delta^2)+y(\lambda-a-b)^2+z(\lambda-a-c)^2=0$$

must pass through $(0, 1, 0)$; the λ-value for B is:

$$\lambda=a+b.$$

The λ-conic is $$(a+b)\phi - 4xk = 0$$

which has as tangent at B

$$x(\gamma + b) = z(a + b)$$

that is, the cuspidal tangent to Q at B, t_B. Thus B and its A-correspondent are given by the conic of the pencil tangent at B to t_B, and the tangent to Θ at B.

For C, symmetrically,

$$\lambda = a + c$$

the λ-conic is tangent to t_C at C

the λ-line is tangent to Θ at C.

To obtain the λ-values corresponding to K_1K_2, another method is more easily handled. It would apply as well as B and C. Since K_1 is given on Θ by a λ-value (i. e. as the point of contact of a λ-line),

$$(h + 2k)/g = \lambda$$

must pass through K_1, the coordinates of which are known. The λ-value obtained is:

$$\lambda = (a\alpha - \beta\gamma - \sigma\delta)/a$$

The λ for K_2 (obtained by changing the sign of σ) is:

$$\lambda = (a\alpha - \beta\gamma + \sigma\delta)/a.$$

For $ABCK_1K_2$ designated as described, α is negative; β, γ are positive; δ, which is less than $b + c$ and therefore less than a, may be assumed positive, for since only the order BAC on ϕ is of importance, BC may be so named that $b > c$. The special case $b = c$ will be discussed at the end of this section. Now $a\alpha - \beta\gamma + \sigma\delta$ is negative, for $(\beta\gamma - a\alpha)^2 - \sigma^2\delta^2$ reduces to

$$4a^4 - 4a^3(b + c) + a^2(b + c)^2 + 3a^2\delta^2 + 2a(b + c)\delta^2.$$

Since $4a^4 - 4a^3(b + c)$ is $-4a^3\alpha$, which is positive, and all the other terms are positive,

$$|\beta\gamma - a\alpha| > |\sigma\delta|$$

so λ_{K_2} is negative. The λ for K_1 is negative and greater than λ_{K_2} in absolute value.

For points on Q in the neighborhood of A, x is positive and k is positive. Since Q is unipartite, consisting of one infinite branch, it lies entirely outside ϕ, (it has ten intersections with ϕ fixed at the cusps and so cannot cross ϕ).

Consequently, for points on Q in the neighborhood of A, ϕ is negative. Now

$$\lambda\phi = 4xk$$

therefore λ for such points is negative.

As λ varies from ∞ at A through negative values, the points of intersection of the λ-conic and λ-line trace out the two branches from the cusp A. The λ-values for both K_1 and K_2 are negative. Further, with $B\,C$ so named that $b > c$, λ_C is negative, for

$$a^2 > 4bc > 4c^2; \qquad |\,a\,| > 2c$$

so $\lambda = a + c$ is negative. The λ for B

$$\lambda_B = a + b = (\delta^2 - \sigma^2 - ac)/a < \delta^2/a$$

may be negative, or positive less than δ^2/a. On each branch from A, therefore, λ passes through negative values, zero, positive values to $\lambda = (b - c)^2/a$ the order being $\lambda_A\,\lambda_{K_1}\,\lambda_{K_2}\,\lambda_C\,\lambda_B\,\lambda_{I_A}$, the two branches connecting at I_A to complete the single circuit. This apparently does not determine on which branch the cusps lie; it may be that all the cusps lie on one branch between A and I_A, and their correspondents lie on the other. The sequence of λ values does show that whatever the order of $B\,B'\,C\,C'$ may be, there exists an arc BC containing no other cusp, which does or does not contain I_A according as $B\,C$ are on the same or opposite branches; therefore B, C are adjacent cusps on Q. But $B\,A\,C$ are any three cusps adjacent in that order on the conic ϕ; $B\,C$ are any two cusps not adjacent on ϕ; the theorem may therefore be stated:

Any two cusps not adjacent on the conic determined by the five cusps are adjacent on the quintic determined by the five cusps.

Since the quintic is unipartite, and there are only two cusps not adjacent to B on ϕ, viz. C and K_2, the order of cusps on Q is $C\,B\,K_2$. Repetition of this argument for every pair of non-adjacent cusps gives the order on Q:— $B\,K_2\,A\,K_1\,C$. Any given λ value, however, gives two points on Q, e. g. B and its A-correspondent B', which are known to be on opposite branches. Hence consideration of the sequence of λ-values determines the following order on Q:

$$B\,C'\,K_2\,K'_1\,A\,K_1\,K'_2\,C\,B'$$

where the primed letters denote A-correspondents of the cusps. But the λ for I_A is greater than λ for B and B'; I_A lies between B and B'. Consequently $B\,C$ are separated by $A\,I_A$ in such a way that on the arc $B\,I_A\,C$ there are no

other cusps.* Since $B\ C$ are any two cusps non-adjacent on ϕ, and A is then the cusp adjacent to both, the same process may be repeated, determining the relative position of the other four inflexions. The resulting order on Q is:

$$A\ (C)\ K_2\ (K_1)\ B\ (A)\ C\ (K_2)\ K_1\ (B)\ A$$

where the bracketed letters denote inflexions associated with the given cusps.

Comparison of the order on Q with the order on ϕ gives the two theorems:

Any two cusps adjacent on Q are not adjacent on the conic ϕ determined by the five cusps.

Any two cusps not adjacent on Q are adjacent on ϕ.

The assumption has been made that b is different from c. The special relations $a = b$ or $a = c$ are precluded by the fact that a is negative, so that $a > b + c$. Consequently, $b = c$ is the only specialization which affects the foregoing argument. The vanishing of $b - c$ does not interfere with the existence of the equations developed, though the system $S_1(\lambda^2)$, while still involving λ^2, degenerates into a system of lines through a point; that is, equation (6) becomes

$$g(\lambda + a - 3b)^2 + 4ax(\lambda - a + 2b) = 0.$$

Only in the geometrical interpretation is the specialization noticeable. In the preceding development, the following changes appear:

The λ-value for Θ is zero, so Θ becomes the line pair $xk = 0$. The double point of the line pair lies on the Q and is therefore the inflexion, since Θ cuts out I_A on Q. The inflexional tangent j_A is the line $l + 4ak$ with $b = c$, given by $\lambda = 0$ in equation (8). The coordinates for I_A reduce to $x : y : z = 0 : -1 : 1$, that is, the common point of $x = 0$, $ax + b(y + z) = 0$. But A_1, the conjugate of A with respect to $S_2(\lambda)$, with coordinates $x : y : z = \delta : -2a : 2a$ becomes $(0, -1, 1)$. In this case, therefore, A_1 is the inflexion I_A, and the tangent to ϕ viz. $g = 0$, which is the line AA_1, passes through the point (xk). The cuspidal tangent at A, t_A, becomes $y - z = 0$. The tangents t_B and t_C become respectively

* Analytical proof may be given that $B\ C$ are separated by $A\ I_A$. In equation (18), call $Px = (T - R)y$ solution I and $Px = (T + R)y$ solution II. Points common to solution I and the λ-conics trace out one branch of Q from A; points of solution II trace the other branch; the two solutions come again to coincidence at I_A. Substitution of coordinates $(0\,1\,0)$, B, gives $T - R = 0$; $\lambda = a + b$ gives $T = 2b(\sigma^2 + ac)$, which is positive; therefore $T - R$ is the desired coefficient, and B lies on branch *I*. The R thus evaluated (for B) is $2b(\sigma^2 + ac)$.

For C, $\lambda = a + c$; $R = 2c(\sigma^2 + ab)$. The desired line is tangent to the λ-conic at C and is therefore of the form $y(a + c) = x(\beta + c)$. Now $\lambda = a + c$ gives $P = (\beta + c)(\delta^2 - 4ac)$, $T + R = (a + c)(\delta^2 - 4ac)$. These coefficients have the desired form, so C lies on branch II. Then $B\ C$ are separated by $A\ I_A$.

$$x(a+b)=z(-a+3b)$$
$$x(a+b)=y(-a+3b).$$

and hence meet on $y-z=0$, that is, on t_A. The equations for t_{K_1} and t_{K_2} reduce to

$$2ax(ab-\sigma^2)+y[2ab^2-\sigma(a-b)-\sigma^2(a+b)]$$
$$+z[2ab^2+\sigma(a-b)-\sigma^2(a+b)]=0$$
$$2ax(ab-\sigma^2)+y[2ab^2+\sigma(a-b)-\sigma^2(a+b)]$$
$$+z[2ab^2-\sigma(a-b)-\sigma^2(a+b)]=0$$

and so meet on t_A. The two points of intersection on t_A

$$(20)\qquad x:y:z=-a+3b:a+b:a+b$$
$$x:y:z=2ab^2-\sigma^2(a+b):a(\sigma^2-ab):a(\sigma^2-ab)$$

are distinct if no further condition is imposed.

Conversely, if a pair of cuspidal tangents t_B, t_C meet on t_A, the condition for concurrence gives

$$(\beta+c)(\alpha+b)=(\gamma+b)(\alpha+c)$$

therefore $\alpha=0$ or $b=c$. With five real cusps $\alpha=0$ is impossible, so $b=c$; then t_{K_1} and t_{K_2} also meet on t_A, and I_A is the point common to $x=0$, $k=0$. If I_A is known to be the point (xk), then $b=c$, and the cuspidal tangents meet as stated. Any one of the three conditions brings about the other two.

If now t_B t_C and t_{K_1} t_{K_2} meet as paired on t_A, and (e. g.) t_A t_{K_1} and t_C t_{K_2} meet on t_B, in other words, if all five cuspidal tangents are concurrent, then $ABCK_1K_2$ may be projected into the vertices of a regular pentagon; for I_A is the point (xk), I_B is the point (AK_1, CK_2). Project the line joining these two inflexions to infinity, and the conic ϕ into a circle. Then g, x, k are parallels, so the arcs AB, BK_1, AC, CK_2 are all equal. Similarly, the tangent to ϕ at B and its parallels give the arcs BA, AC, K_1K_2, K_1B all equal, so $ABCK_1K_2$ is a regular pentagon and the original five cusps were such a set as could be projected into its vertices. The five inflexions in this symmetrical case are at infinity.

Such concurrence of the cuspidal tangents may be obtained from the case $b=c$, $\alpha<0$, by imposition of another condition on a, b, such as coincidence of the two points (20) on $y-z=0$. Proportionality of the respective x and y coordinates gives (for $b=c$, $\sigma^2=a(a-4b)$)

$$a(-a+3b)(a-5b)=[2b^2-(a-4b)(a+b)](a+b).$$

The resulting condition is:

$$a=b(2\pm\sqrt{5});\quad\text{so}\quad a=b(2+\sqrt{5})\ \text{since}\ a>2b.$$

This is a high degree of specialization and results in much simplification of the framework of related lines and points (see Fig. 2); as one result, the linear constructions about to be given are not available.

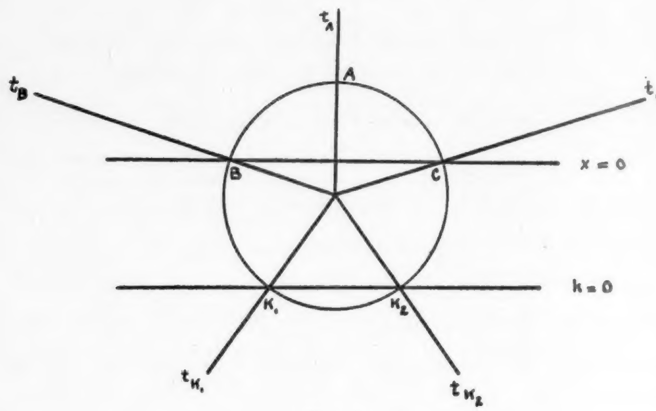

FIG. 2.

The Appearance of the Quintic Curve. The curve is unipartite, composed of one infinite branch. It has five cusps and five inflexions occurring alternately, each cusp associated with the inflexion symmetrically placed with respect to it and the remaining four cusps. The parametric treatment has given this information. A line having only one real intersection with the curve certainly exists, for the line joining two cusps adjacent on ϕ meets the curve in one other point, P. Rotate this line about P through a very small angle so that its intersections with the branches from the cusps become imaginary; since the curve lies entirely outside ϕ, such rotation about an external point is always possible. The line in its new position has only one real intersection with Q. Projection of this line to infinity gives the type of the curve with only one real asymptote. Figures 3 and 4, obtained by quadratic transformation of a quartic with two cusps, in- and circumscribed to the triangle of reference ABC, show this form and also the form with five real asymptotes.

The inflexional tangent j_A at I_A divides ϕ into two arcs, on one of which A lies, while $B\,C\,K_1K_2$ all lie on the other arc. The equation of j_A is (10a). Let j_A cut ϕ in points $J_1\,J_2$; the lines AJ_1, AJ_2 are then:

$$y^2\left[\frac{\sigma^2+ac}{a}\right]^2+yz\left[3\delta^2-4a\alpha+\frac{(\sigma^2+ac)^2}{a^2}+\frac{(\sigma^2+ab)^2}{a^2}\right]+z^2\left[\frac{\sigma^2+ab}{a}\right]^2=0.$$

The discriminant here is:

$$\left[3\delta^2 - 4a\alpha + \frac{(\sigma^2+ac)^2}{a^2} + \frac{(\sigma^2+ab)^2}{a^2}\right]^2 - 4\,\frac{(\sigma^2+ab)^2\,(\sigma^2+ac)^2}{a^4}$$

$$= (3\delta^2 - 4a\alpha)^2 + 2(3\delta^2 - 4a\alpha)\left[\frac{(\sigma^2+ac)^2}{a^2} + \frac{(\sigma^2+ab)^2}{a^2}\right]$$

$$+ \left[\frac{(\sigma^2+ac)^2}{a^2} - \frac{(\sigma^2+ab)^2}{a^2}\right]^2.$$

The terms are all positive; the roots are therefore real. The coefficients of the quadratic are all positive; the lines AJ_1, AJ_2 therefore lie outside the triangle, and J_1J_2 separate A from BCK_1K_2 on ϕ.

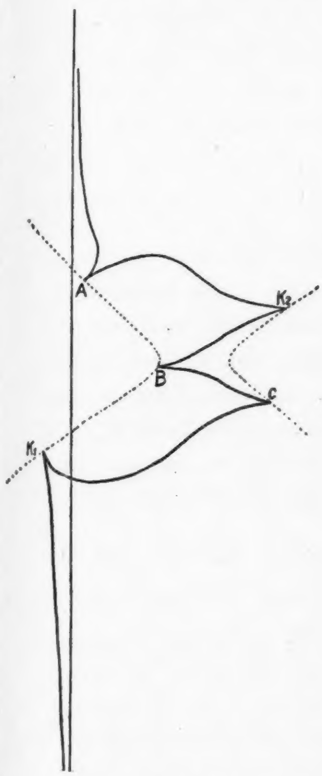

FIG. 3.

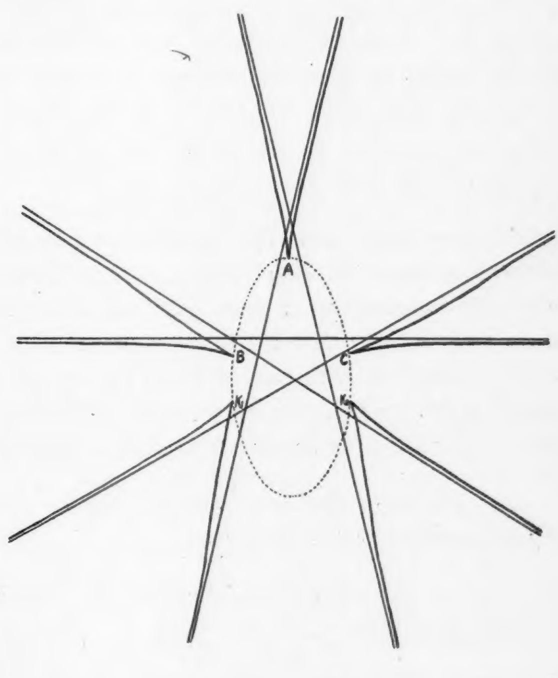

FIG. 4.

The curve lies in such a way that tangents to ϕ at the cusps have each three real intersections with Q elsewhere. For: $g=0$ has intersections with Q given by $y^2=0$ and the cubic

$$y^3\delta(a+3\alpha) + y^2x(9\delta^2+4a\alpha) + yx^2(24a\delta) + 16a^2x^3 = 0.$$

The substitution $4ax/y = w - 2\delta$ gives

$$w^3 + w(4a\alpha - 3\delta^2) + 2\delta^2(a^2 - \sigma^2) = 0.$$

The derived equation has two real roots w_1, w_2; the cubic expression evaluated at $-\infty, w_1, w_2, +\infty$ exhibits three changes of sign; three real roots exist; therefore $g = 0$ has three real intersections with Q not at A.

Since the Plücker numbers for the quintic with five cusps, or pentoid, as it may be called, are $m = n = 5$; $\kappa = i = 5$, the reciprocal curve is another pentoid. For this curve, corresponding theorems may be stated. The conic of the cusps, ϕ, becomes χ the conic of the five inflexional tangents, $j_A\, j_B\, j_{K_1}\, j_{K_2}\, j_C$, with points of contact on χ in the order given. The order of points of contact on Q is then (A) (K_2) (B) (C) (K_1). From the point of contact of j_A with χ, three real tangents may be drawn. Through the cusp A there pass two tangents to χ which cut off an arc of χ to which $j_B\, j_C\, j_{K_1}\, j_{K_2}$ are tangents and an arc to which j_A alone is tangent. Since any pentoid has an equation of the type of equation (4), these properties, which relate only to order on the curve and reality of elements, characterize the original pentoid also.

Constructions. 1). The cuspidal tangent to Q at A is the polar of A_1 (the conjugate of A with respect to $S_2(\lambda)$), with respect to ϕ. Now A_1 on g may be constructed linearly; (AA_1 are harmonic with respect to the points (xk) on g); then a linear construction for t_A follows. Similarly, t_B is the polar of B_1 with respect to ϕ, (where B_1 is the conjugate of B with respect to the pencil of conics through ACK_1K_2). So the five cuspidal tangents are obtained by linear constructions from the five given cusps.

2). The five inflexional tangents may be constructed linearly, with the help of certain intermediate lines.

$p \equiv \Sigma x = 0$ is the Pascal line for ABC inscribed in ϕ; it may therefore be constructed linearly.

$h = (c - a)y + (b - a)z$ is then known: for p meets k in a point P on the line AP:

$$(b - a)y + (c - a)z = 0;$$

then h is the harmonic conjugate of AP with respect to $y \pm z = 0$;

$$\begin{aligned}[(b - a)y + (c - a)z] + [(c - a)y + (b - a)z] &= (y + z)\cdot(\alpha - a)\\ [(b - a)y + (c - a)z] - [(c - a)y + (b - a)z] &= (y - z)\delta.\end{aligned}$$

Now h and k are known; $h+2k$ is the line through their common point and through A_1; $h+2k$ may be constructed.

The pencil of lines joining A_1 to the λ-points on Θ

$$\lambda g-(h+2k)=0$$

is projective with the pencil of polars of A with respect to $S_2(\lambda)$, viz.,

$$\lambda t_A-4\delta(by+cz)=0.$$

In these pencils the rays correspond as follows:

(A_1 pencil)		(A pencil)
$g(\lambda=\infty)$	corresponds to	t_A
$h+2k(\lambda=0)$	corresponds to	$by+cz=0$
$A_1B\ \ (\lambda=a+b)$	corresponds to	polar of $A_1 w.r.$ to conic of $S_2(\lambda)$ tangent at B to t_B.

All three lines in the A_1 pencil are constructed; t_A, t_B are drawn; $by+cz=0$ is the line joining A to the common point of $x=0$, $k=0$; the polar of A_1 with respect to a conic through four given points, with a given tangent at one of them, may be constructed linearly. Therefore: three pairs of corresponding rays in the $A\,A_1$ pencils are drawn and the correspondent to any other ray may be constructed. Since g as a ray of the A pencil corresponds to $\lambda=\delta^2/a$,

$$\begin{aligned}(\delta^2/a)t_A-4\delta(by+cz) &= (\delta/a)\{y[\delta(2a+\delta)-4ab]+z[-\delta(2a-\delta)-4ac]\}\\ &= (\delta/a)(y+z)(\sigma^2-a^2),\end{aligned}$$

its corresponding ray (which may be linearly constructed), determines I_A on Θ; j_A, tangent to a conic given by five points, at a known point on it, is then given by linear construction. For B, a similar process is performed, with the system $S_2(\mu)$ of conics through ACK_1K_2 as basis, and the conic of $S_2(\mu)$ through B_1 in place of Θ. Thus all five inflexions and the inflexional tangents are obtained by linear construction from the five given cusps.*

3). The intermediate lines necessary in this construction give other points on the quintic.

* Any of the given quantities a, b, c, σ may be irrational, but the parameters and coordinates of all the points in question are built up from these quantities by rational operations.

The fifth point of intersection on x, k respectively (given by the conic of $S_2(\lambda)$ corresponding to $\lambda = 0$), on the line $l + 4ak = 0$ may be constructed, for $l + 4ak = 0$ is the tangent to Θ at the point given by $h + 2k$ (through A_1); a linear construction gives $l + 4ak$. The fifth point on all lines joining two cusps may be obtained by repetition of this process at B, C, K_1, K_2. There are ten such lines, therefore ten points are given.

The A-correspondent to B lies on the conic of $S_2(\lambda)$ tangent at B to t_B, and on the line tangent to Θ at B. This line and its intersection with the conic in question may be constructed linearly; B' is therefore found. Four such points are given for B (correspondents with respect to $A\,C\,K_1\,K_2$). Similarly, four correspondents for each cusp may be found—twenty points in all. The series of points obtained by the linear constructions in (3) may be extended in this way indefinitely.

BRYN MAWR, PA., 1925.

The Convergence of General Means and the Invariance of Form of Certain Frequency Functions.

By Edward L. Dodd.

Introduction. The problem of choosing for given statistical data the most suitable mean or average offers many difficulties. Perhaps for some time, the intuition or the judgment of the investigator must be relied upon chiefly. However, it seems possible to work out to some extent the relations between frequency functions and means so that eventually the choice may be more guided.

To illustrate, suppose that a distribution presents itself that can be fitted admirably by the function or curve

$$y = \frac{x^{-2/3}}{3(2\pi)^{1/2}} e^{-(x^{1/3}-4)^2/2}, \tag{1}$$

discussed by Rietz.* In obtaining (1) the assumption made—that diameters are normally distributed corresponding to the volumes under consideration—suggests that a very suitable mean for (1) is

$$M = [(1/n) \sum m_i^{1/3}]^3, \qquad (i = 1, 2, \cdots, n). \tag{2}$$

Moreover, when for independent measurements m_i, the probability that $x < m_i < x + dx$ is $y\,dx$, as given by (1), then the probability that $x < M < x + dx$ is $v\,dx$, where

$$v = \frac{n^{1/2} x^{-2/3}}{3(2\pi)^{1/2}} e^{-n(x^{1/3}-4)^2/2}. \tag{3}$$

And this expression not only maintains the essential form of (1), but it shows that with increasing n there is a probability bordering on certainty that M will converge to 64, a value that the underlying assumption suggests as the "normal value" or "true value" of m.

This mean given by (2) is one of a general class of means † that can be written in the form

* "Frequency Distributions Obtained by Certain Transformations of Normally Distributed Variates," *Annals of Mathematics*, Ser. 2, Vol. 23, (1922), pp. 292-300—see p. 295. Primes used by Rietz have been dropped.

† For a more general mean involving weights, see Dodd, "Functions of Measurements under General Laws of Error," *Skandinavisk Aktuarietidskrift*, Vol. 5 (1922), pp. 133-158. See p. 141.

$$(4) \qquad M = G[(1/n) \sum_1^n f_i(m_i)],$$

where each $f_i(u)$ is a continuous function of u, univariant over the interval in which the m_i appear, $v = 1/n \Sigma f_i(u) = F(u)$, $u = G(v)$. Setting $f_i(u) = u$, $\log u$, $1/u$, u^2, $u^{1/3}$, we obtain the arithmetic mean, the geometric mean, the harmonic mean, the root-mean-square, and the cube of the mean cube-root in (2), respectively.

The purpose of this paper is to exhibit certain relations that connect arbitrary frequency functions with the general mean M in (4), relative to the convergence of M to a value a, which may be taken as the true value, and relative also to the permanence or essential invariance of the form of the frequency function, as illustrated by (1) and (3).

1. *Theorems on General Means and Frequency Functions.*

THEOREM I. *Let the probability that a measurement m_i will take on a value less than u be $\Phi(u)$, where $\Phi(\alpha) = 0$ and $\Phi(\beta) = 1$. And suppose that in the interval from α to β, finite or infinite, $\Phi(u)$ is an increasing function, and—except possibly for a finite number of values of u in any finite interval—has a positive derivative $\Phi'(u)$. Let $\Psi(v)$ be a second arbitrary cumulative frequency function, defined in the interval from α' to β'* with *corresponding restrictions, and such that*

$$(5) \qquad \Psi(a) = \Phi(a)$$

$$(6) \qquad \int_{\alpha'}^{\beta'} v\,\Psi'(v)\,dv = a, \qquad \int_{\alpha'}^{\beta'} v^2\,\Psi'(v)\,dv < K \text{ (finite)}.$$

Then the continuous increasing function

$$(7) \qquad v = f(u)$$

with inverse $u = g(v)$, constructed by setting*

$$(8) \qquad \Psi(v) = \Phi(u),$$

is such that if for n independent measurements m_i,

$$(9) \qquad M = g[(1/n) \Sigma f(m_i)],$$

and if ϵ and η are arbitrarily small positive numbers, there is a probability greater than $1 - \eta$, when n is sufficiently large, that

$$(10) \qquad |M - a| < \epsilon.$$

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* Dodd, "The Frequency Law of a Function of One Variable," *Bulletin of the American Mathematical Society*, Vol. 31 (1925), pp. 27-31. See p. 30.

Proof. Equation (8) may be written in the differential form

$$\Psi'(v)\,dv = \Phi'(u)\,du, \tag{11}$$

and each member may be regarded as the probability that both the following equivalent inequalities will be satisfied:

$$u < m_i < u + du, \qquad v < f(m_i) < v + dv, \tag{12}$$

noting that $f(u)$ is an increasing function.

Then, with $\epsilon' > 0$, $\eta > 0$, it is possible to take n large enough—on account of (6) and the Tchebycheff Theorem—so that there will be a probability greater than $1 - \eta$ that

$$a - \epsilon' < (1/n) \sum_1^n f(m_i) < a + \epsilon'. \tag{13}$$

But, from (5), (7), and (8), it follows that

$$f(a) = a, \qquad g(a) = a. \tag{14}$$

And thus (13) may be written, on account of (9),

$$a - \epsilon_1 = g(a - \epsilon') < M < g(a + \epsilon') = a + \epsilon_2, \tag{15}$$

where ϵ_1 and ϵ_2 approach zero with ϵ', since $g(v)$ is a continuous function. This establishes (10).

Theorem II. *Let the probability that a measurement m_i will take on a value less than u be $\Phi_i(u)$, where in the interval $\alpha_i \leqq u \leqq \beta_i$, each $\Phi_i(u)$ is under the restrictions of Theorem I. Suppose, furthermore, that the measurements are independent, and that there exist constants a, δ, and k, so that if $a - \delta \leqq u \leqq a + \delta$,*

$$\Phi_i'(u) > k > 0, \qquad (i = 1, 2, \cdots, n). \tag{16}$$

Then there exist continuous increasing functions $f_i(u)$, defined in (α_i, β_i), so that if we set

$$v = F(u) = (1/n) \sum_1^n f_i(u), \qquad u = G(v), \tag{17}$$

$$M = G[(1/n) \sum f_i(m_i)], \tag{18}$$

and if we take two positive numbers ϵ and η small at pleasure, there is a probability greater than $1 - \eta$, when n is sufficiently large, that

$$| M - a | < \epsilon. \tag{19}$$

Proof. We can select in an infinite number of ways functions $\Psi_i(v)$, so that if $a - \delta \leqq v \leqq a + \delta$,

$$\Psi_i'(v) < k, \qquad (i = 1, 2, \cdots, n). \tag{20}$$

functions conforming, moreover, to the requirements of Theorem I, in particular, (5), (6), and (8); since these merely require that the area under Ψ_i' up to the point v shall be equal to that under $\Phi'(u)$ up to the point u —where u and v reach a simultaneously—and that the first moment or center of gravity for Ψ_i' shall be equal to a, and the second moment shall be finite. Thus we can write (13) and (14) with f replaced by f_i, g by G. Then, from (16), (17), and (20), it follows that when $a - \delta \leqq v \leqq a + \delta$, $G'(v) < 1$. Hence, a fortiori, (13) is valid if the middle member is replaced by M; and this leads to (19).

THEOREM III. *Let the probability that a measurement m_i will take on a value less than u be $\Phi_i(u)$, where in the interval from α_i to β_i, $\Phi_i(u)$ is under the restrictions stated in Theorem II. Suppose, furthermore, that*

$$\Phi_i(a) = \tfrac{1}{2}, \qquad (i = 1, 2, \cdots, n). \tag{21}$$

Take

$$w = \Theta(y) = (1/\pi^{1/2}) \int_{-\infty}^{y} e^{-t^2} dt, \qquad y = \Theta^{-1}(w), \tag{22}$$

$$f_i(u) = a + \Theta^{-1}[\Phi_i(u)], \qquad v = f(u) = (1/n) \sum_{1}^{n} f_i(u), \tag{23}$$

$$u = g(v), \qquad M = g[(1/n) \sum f_i(m_i)]. \tag{24}$$

Then the probability that

$$u_1 < M < u_2, \tag{25}$$

when $a - \delta \leqq u_1 \leqq u_2 \leqq a + \delta$, is given by

$$P(u_1, u_2) = \Theta\{n^{1/2}[f(u_2) - a]\} - \Theta\{n^{1/2}[f(u_1) - a]\}, \tag{26}$$

whether n be large or small. Moreover, if ϵ and η are arbitrarily small positive numbers, there is a probability greater than $1 - \eta$ when n is sufficiently large, that

$$| M - a | < \epsilon. \tag{27}$$

Proof. Let us write (23) in the form

$$\Theta[f_i(t) - a] = \Phi_i(t), \tag{28}$$

and seek the probability that

$$s < f_i(m_i) < s + ds. \tag{29}$$

Take

(30) $$t = g_i(s), \qquad s = f_i(t),$$

and as equivalent to (29) write

(31) $$t < m_i < t + dt.$$

From (22), (28), and (30), the probability for (31) and thus (29) is

(32) $$\Phi'(t)\,(dt) = (1/\pi^{1/2})\, e^{-(s-a)^2} ds.$$

Hence, by a well-known theorem,* the probability that

(33) $$s < (1/n) \sum_1^n f_i(m_i) < s + ds$$

is

(34) $$(n^{1/2}/\pi^{1/2})\, e^{-n(s-a)^2} ds.$$

We may now choose this s as identical with the v in (24), and write

(35) $$(n^{1/2}/\pi^{1/2})\, e^{-n(v-a)^2} dv$$

as the probability that

(36) $$u < M < u + du.$$

In (35) we may set $v = f(u)$ from (23), and integrate between u_1 and u_2 as required by (25), using a simple change of variable; and this leads to (26) on account of (22).

Furthermore, if we take $0 < \epsilon \leqq \delta$, we may conclude from (16), (21), (22), (23), that

(37) $$\Phi_i(a - \epsilon) < \tfrac{1}{2} - k\epsilon, \qquad \Phi_i(a + \epsilon) > \tfrac{1}{2} + k\epsilon,$$

(38) $$f(a - \epsilon) < a + \Theta^{-1}(\tfrac{1}{2} - k\epsilon), \qquad f(a + \epsilon) > a + \Theta^{-1}(\tfrac{1}{2} + k\epsilon).$$

and thus, from (26), $P(a - \epsilon, a + \epsilon)$ approaches unity with increasing n.

Corollary. Under the conditions of the theorem, it is possible to take n large enough so that if ξ is a preassigned positive number, the probability that

(39) $$M < z$$

is given by

(40) $$p(z) = \Theta\{n^{1/2}[f(z) - a]\} + \xi', \qquad 0 \leqq \xi' \leqq \xi.$$

Proof. In (37) and (38) take $\epsilon = \delta$, and determine n so that in (26), $P(a - \delta, a + \delta) > 1 - \xi$. Then in (26) take $u_2 = z$, $u_1 = a - \delta$.

2. *Illustration.* Suppose that for independent measurements m_i necessarily positive, the probability that $m_i < u$ is given by

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* Czuber, *Wahrscheinlichkeitsrechnung*, Vol. I, third edition (1914), p. 304.

$$(41) \qquad \Omega(u, h) = \Phi(u) = (h/\pi^{1/2}) \int_0^u e^{-(h \log t/a)^2} \, dt/t,$$

and that M is the geometric mean

$$(42) \qquad M = (m_1 m_2 \cdot \cdot \cdot m_n)^{1/n}.$$

Then the probability that $M < z$ is given by

$$(43) \qquad \Omega(z, n^{1/2} h) = \Theta(n^{1/2} h \log z/a),$$

where Θ is the common probability function in (22). To show this, use in (23),

$$(44) \qquad v = f(u) = a + h(\log u - \log a).$$

3. *Summary and Discussion.* Under rather general conditions, Therem II establishes the existence of means of n independent measurements converging with asymptotic certainty to the true value as n increases.

With an added requirement that the probability of a negative error shall be 1/2, it is shown in Theorem III that a mean may be constructed subject to a frequency law depending in a simple manner upon the usual probability integral. The computations here, indeed, involve for the most part simple arithmetic operations, to be supplemented by direct and inverse use of a probability table. This table may be based as well upon $e^{-t^2/2}$ as upon e^{-t^2}; since a change of t to $t\sqrt{2}$ in (22) would not involve any change in the reasoning. The work is especially simple if all the measurements are subject to the same frequency law.

The above condition that the probability of a negative error shall be 1/2 may be illustrated by the lateral distribution of errors of a golf ball aimed at the top of a mound with slopes which may be different on the two sides. It does not require any symmetry for the frequency curve.

Theorem III would seem to be of special interest in the case where the frequency functions $f_i(u)$ in (23) undergo a cyclic change yielding $f(u)$ as an average.

That this paper does not clear up all difficulties may be illustrated by referring to Rietz's example in (1). Under this law the arithmetic mean tends toward 76, thus creating an error of 12 if 64 is regarded as the normal value. But, if in the determination of the parameter in (1) by fitting the data, an error is made so that the parameter is written 4.236 instead of 4, then the cube of the mean cube-root, as given by (2) tends towards the same 76. An error of 12 for 64 is bad, and almost forces us to abandon the arithmetic mean. But caution must be exercised in choosing a substitute when the number of measurements is too scanty to establish within narrow bounds the parameters of the associated frequency curve.

On the Interpolatory Properties of a Linear Combination of Continuous Functions.

By D. V. Widder.

1. *Introduction.* In his treatment of the mean-value theorems belonging to a linear differential equation, G. Pólya* obtained a condition on a set of functions

$$u_i(x) \qquad (i=1,2,3,\cdots,n) \tag{1}$$

which insured that any linear combination of them should enjoy the more important interpolatory properties of a polynomial of degree less than n. He designated the condition as the *property* W. We give the definition with a slight modification.†

Definition. If it is possible to determine constants c_{ij} *so that the* functions

$$h_i(x) = \sum_{j=1}^{n} c_{ij}\, u_j(x), \qquad (i=1,2,3,\cdots,n)$$

satisfy the inequalities

$$\begin{vmatrix} h_1(x) & h_2(x) & \cdots & h_k(x) \\ h_1'(x) & h_2'(x) & \cdots & h_k'(x) \\ \cdot & \cdot & \cdot & \cdot \\ h_1^{(k-1)}(x) & h_2^{(k-1)}(x) & \cdots & h_k^{(k-1)}(x) \end{vmatrix} > 0, \qquad (k=1,2,3,\cdots,n)$$

in an interval (a,b), *then the set* (1) *possesses the property* W *in that interval. The functions* $u_i(x)$ *are assumed to have continuous derivatives of order* n *throughout the interval.*

It was shown, for example, that this condition on the functions (1) is necessary and sufficient that no linear combination of them, except the identically vanishing one, shall vanish n times in (a,b).

This condition has the disadvantage that it demands the existence of n

* "On the Mean-Value Theorem Corresponding to a Given Linear Homogeneous Differential Equation," *Transactions of the American Mathematical Society*, Vol. 24 (1922), pp. 312-324.

† This modification is made in order to make the condition independent of the differential equation treated by Pólya.

continuous derivatives for the functions in question. In certain problems of interpolation these functions are known only to be continuous. A case in point is that of the generalized problem of Tschebycheff.* It would be desirable then to impose a condition on the functions $u_i(x)$ that will not involve the existence of the derivatives, but which will guarantee that any linear combination of these functions shall resemble a polynomial in its interpolatory properties.

It is the purpose of the present paper to obtain such a condition and to apply it to the solution of several problems.

2. *The Property V.* Consider a set of functions (1), all continuous in an interval $a \leqq x \leqq b$ of the x-axis.

Definition. If for every value d, $a < d < b$, it is possible to determine constants c_{ij} so that the functions

$$h_i(x) = \sum_{j=1}^{n} c_{ij}\, u_j(x), \qquad (i = 1, 2, 3, \cdots, n)$$

satisfy the inequalities

$$V_k(x) = V[h_1(x), h_2(x), \cdots, h_k(x)] =$$

$$\begin{vmatrix} h_1(x) & h_2(x) & \cdots & h_k(x) \\ h_1(x+\delta) & h_2(x+\delta) & \cdots & h_k(x+\delta) \\ \cdot & \cdot & \cdot & \cdot \\ h_1(x+\overline{k-1}\,\delta) & h_2(x+\overline{k-1}\,\delta) & \cdots & h_k(x+\overline{k-1}\,\delta) \end{vmatrix} > 0$$

$$(k = 1, 2, 3, \cdots, n)$$

for every value of x in the interval $a \leqq x \leqq d$ and for every value of δ in the interval $0 < \delta \leqq (b-d)/n$, then the set (1) is said to possess the property V in (a, b).

As an example of such a set of functions take

$$h_i(x) = u_i(x) = x^{i-1}, \qquad (i = 1, 2, 3, \cdots, n).$$

Following the usual notation we set

$$\Delta^{\delta}_{n} f(x) = f(x+n\delta) - nf(x+\overline{n-1}\,\delta) + \frac{n(n-1)}{2!} f(x+\overline{n-2}\,\delta) + \cdots$$
$$+ (-1)^n f(x), \quad \Delta^{\delta}_{1} f = \Delta f.$$

* See § 6.

The functions $V_k(x)$ may then be written

$$V_k(x) = \begin{vmatrix} h_1 & h_2 & \cdots & h_k \\ \Delta^\delta{}_1 h_1 & \Delta^\delta{}_1 h_2 & \cdots & \Delta^\delta{}_1 h_k \\ \cdot & \cdot & \cdot & \cdot \\ \Delta^\delta{}_{k-1} h_1 & \Delta^\delta{}_{k-1} h_2 & \cdots & \Delta^\delta{}_{k-1} h_k \end{vmatrix}.$$

Now

$$\Delta^\delta{}_j\, x^{k-1} = \begin{cases} \delta^{k-1}(k-1)\,!, & j = k-1 \\ 0, & j > k-1 \end{cases}$$

Hence it is readily verified that

$$V_k(x) = (k-1)\,!\,(k-2)\,!\cdots 2\,!\;\delta^{(k-1)k/2}.$$

This is positive for any positive value of δ and is independent of x, so that the property V holds in any interval of the x-axis.

It is to be noted that the functions $h_i(x)$ may depend on d. As an example take the set of functions

$$u_1(x) = \sin x, \qquad u_2(x) = \cos x.$$

This set possesses the property V in the interval $(a, a+\pi)$, a being arbitrary. For, take

$$V_1(x) = h_1(x) = \sin[x + (\pi - a - d)/2]$$
$$h_2(x) = -\cos[x + (\pi - a - d)/2], \qquad V_2(x) = \sin\delta.$$

$V_1(x)$ and $V_2(x)$ are positive in the interval $a \leqq x \leqq d < b$ for all values of d. Clearly it would be impossible to find V_1 independent of d, since every linear combination of the two given functions has at least one zero in any closed interval of length π. More generally it may be shown that the property V holds for the set

$$\sin x,\ \cos x,\ \sin 2x,\ \cos 2x,\ \cdots,\ \sin nx,\ \cos nx$$

in the same interval. If a constant not zero is added to the set, the resulting set will enjoy the property in any interval of length not greater than 2π.

It will appear later that the property W implies the property V. That the converse is not true becomes evident by an example. Let $\phi(x)$ be any continuous monotonic increasing function (not constant in any interval however small) in an interval (a, b). Consider the set of functions

$$(2) \qquad h_i(x) = [\phi(x)]^{i-1}, \qquad (i = 1, 2, 3, \cdots, n).$$

Then

$$V_k(x) = \begin{vmatrix} 1 & \phi(x) & [\phi(x)]^2 & \cdots & [\phi(x)]^{k-1} \\ 1 & \phi(x+\delta) & [\phi(x+\delta)]^2 & \cdots & [\phi(x+\delta)]^{k-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \phi(x+\overline{k-1}\,\delta) & [\phi(x+\overline{k-1}\,\delta)]^2 & \cdot & [\phi(x+\overline{k-1}\,\delta)]^{k-1} \end{vmatrix}$$

This is a Vandermonde determinant, and may be expanded as follows:

$$\begin{aligned} V_k(x) = {} & [\phi(x+\overline{k-1}\,\delta) - \phi(x+\overline{k-2}\,\delta)] \\ & [\phi(x+\overline{k-1}\,\delta) - \phi(x+\overline{k-3}\,\delta)] \cdots \\ & [\phi(x+\overline{k-1}\,\delta) - \phi(x)] \cdots [\phi(x+\delta) - \phi(x)]. \end{aligned}$$

This function is defined and positive in the interval $a \leqq x \leqq d < b$ for any positive δ at most equal to $(b-d)/n$. Each factor of V_k is positive since $\phi(x)$ is monotonic increasing. Hence the property V holds in the interval (a, b). If $\phi(x)$, does not have a continuous derivative, the definition of the property W is not applicable. Another instructive example is obtained by setting

$$\phi(x) = x^3$$

in the set (2). The resulting set is one for which the property W does not hold in any interval including the origin, since the function x^{3n-3} vanishes more than $(n-1)$ times at the origin. Yet the property V, on the other hand, is seen to hold in any interval since x^3 is monotonic increasing.

The property V insures the linear independence of the functions (1). For, if the functions were linearly dependent, constants c_i, not all zero, would exist such that

$$\begin{gathered} c_1u_1(x) + c_2u_2(x) + \cdots + c_nu_n(x) = 0 \\ c_1u_1(x+\delta) + c_2u_2(x+\delta) + \cdots + c_nu_n(x+\delta) = 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ c_1u_1(x+\overline{n-1}\,\delta) + c_2u_2(x+\overline{n-1}\,\delta) + \cdots + c_nu_n(x+\overline{n-1}\,\delta) = 0 \end{gathered}$$

for all δ sufficiently small. Hence we should have

$$V[u_1, u_2, \cdots, u_n] \equiv 0.$$

But

$$V[h_1, h_2, \cdots, h_n] = \begin{vmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{vmatrix} V[u_1, u_2, \cdots, u_n]. \tag{3}$$

Since $V[h_1, h_2, \cdots, h_n] > 0$, it follows that neither factor on the right-hand side of this equation can vanish. The contradiction shows that the set (1) is a linearly independent set.

The following remark will be useful in subsequent work. If the property V holds for the set (1) in the interval (a, b), then the set of functions

$$h_1(x),\ h_2(x),\ \cdots,\ h_k(x), \qquad (k \leqq n)$$

corresponding to the constant d, also possesses the property in the interval (a, d). For in this case the constants c_{ij} of the definition may be taken as

$$c_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j. \end{cases}$$

The functions

$$V[h_1],\quad V[h_1, h_2],\quad \cdots,\quad V[h_1, h_2, \cdots, h_k]$$

are all positive in $a \leqq x \leqq d$, and hence also in any sub-interval.

3. *The Operator L^δ.* We now define a linear difference operator L^δ by means of the following expression

$$L^\delta f(x) = V[u_1, u_2, u_3, \cdots, u_n, f].$$

If $f(x)$ is continuous in the closed interval (a, b), then $L^\delta f(x)$ is defined and continuous in the closed interval $(a, b - n\delta)$, supposing $\delta \leqq (b - a)/n$. It is well known* that a linear homogeneous difference operator of order n can be expressed by n successive applications of the operator Δ. We give a new proof of this fact which seems to be simpler than those heretofore given.

Consider the determinant $V[h_1, h_2, \cdots, h_n, f]$ and its adjoint determinant. Form the following minor from the adjoint:

$$H = \begin{vmatrix} V[h_1(x+\delta), \cdots, h_{n-1}(x+\delta), f(x+\delta)] & V[h_1(x+\delta), h_2(x+\delta), \cdots, h_n(x+\delta)] \\ V[h_1(x), \cdots, h_{n-1}(x), f(x)] & V[h_1(x), h_2(x), \cdots, h_n(x)] \end{vmatrix}.$$

The four elements of this determinant, H, are the minors of $h_n(x)$, $f(x)$, $h_n(x + n\delta)$, and $f(x + n\delta)$ in the determinant $V[h_1, h_2, \cdots, h_n, f]$. By

* See, for example, Guldberg and Wallenberg, *Theorie der Linearen Differenzengleichungen*, p. 76.

use of the familiar theorem * about the minors of the adjoint determinant we obtain

$$H = V[h_1(x+\delta), h_2(x+\delta), \cdots, h_{n-1}(x+\delta)] \qquad V[h_1(x), h_2(x), \cdots, h_n(x), f(x)].$$

Simple computation shows that †

$$\Delta \frac{V[h_1, h_2, \cdots, h_{n-1}, f]_x}{V[h_1, h_2, \cdots, h_n]_x} = \frac{H}{V[h_1, h_2, \cdots, h_n]_x \, V[h_1, h_2, \cdots, h_n]_{x+\delta}}.$$

This leads at once to the formula

$$(4) \quad \Delta \frac{V[h_1, h_2, \cdots, h_{n-1}, f]_x}{V[h_1, h_2, \cdots, h_n]_x} = \frac{V[h_1, h_2, \cdots, h_{n-1}]_{x+\delta} \, V[h_1, h_2, \cdots, h_n, f]_x}{V[h_1, h_2, \cdots, h_n]_x \, V[h_1, h_2, \cdots, h_n]_{x+\delta}}.$$

By successive applications of this formula we arrive at the desired result. First note that $L^\delta f$ may be written as follows:

$$L^\delta f = V[u_1, u_2, \cdots, u_n, f] = KV[h_1, h_2, \cdots, h_n, f], \quad K = 1/|c_{ij}|.$$

From formula (4) we obtain directly

$$L^\delta f = K \frac{V[h_1, h_2, \cdots, h_n]_x \, V[h_1, h_2, \cdots, h_n]_{x+\delta}}{V[h_1, h_2, \cdots, h_{n-1}]_{x+\delta}} \Delta \frac{V[h_1, h_2, \cdots, h_{n-1}, f]_x}{V[h_1, h_2, \cdots, h_n]_x}.$$

Applying (4) to $V[h_1, h_2, \cdots, h_{n-1}, f]$, we obtain

$$L^\delta f = K \frac{V_n(x+\delta)\, V_n(x)}{V_{n-1}(x+\delta)} \Delta \frac{V_{n-1}(x+\delta)\, V_{n-1}(x)}{V_n(x)\, V_{n-2}(x+\delta)} \Delta \frac{V[h_1, h_2, \cdots, h_{n-2}, f]_x}{V[h_1, h_2, \cdots, h_{n-1}]_x}.$$

Now apply (4) to $V[h_1, h_2, \cdots, h_{n-2}, f]_x$, and continue the process. The result is

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* See, for example, M. Bôcher, *Introduction to Higher Algebra*, p. 31.

† For brevity we write

$$V[h_1(x), h_2(x), \cdots, h_n(x), f(x)] = V[h_1, h_2, \cdots, h_n, f]_x.$$

$$(5) \qquad L^\delta f = K \frac{V_n(x+\delta)\, V_n(x)}{V_{n-1}(x+\delta)} \;\Delta\; \frac{V_{n-1}(x+\delta)\, V_{n-1}(x)}{V_n(x)\, V_{n-2}(x+\delta)}$$

$$\Delta \cdots \Delta\; \frac{V_1(x+\delta)\, V_1(x)}{V_2(x)\, V_0(x+\delta)} \;\Delta\; \frac{f(x)}{V_1(x)} \;.$$

Here $V_0(x) = 1$. The validity of this formal work depends, of course, on the non-vanishing of the functions $V_i(x)$, $i = 1, 2, \cdots, n$.

3. *The Mean-Value Theorem.*

THEOREM I. *If the function $f(x)$, continuous in the interval $a \leqq x \leqq b$, vanishes at the $(n+1)$ distinct points x_i*

$$a \leqq x_1 < x_2 < \cdots < x_{n+1} < b,$$

and if the set (1) *enjoys the property V in (a, b), then*

$$L^\delta f(\xi) = 0, \qquad x_1 < \xi < x_{n+1} - n\delta$$

for any value of δ satisfying the relations

$$(6) \qquad 0 < \delta < x_{i+1} - x_i, \qquad (i = 1, 2, \cdots, n) \;.$$

$$0 < \delta \leqq \frac{b - x_{n+1}}{n} \;.$$

To establish this theorem we make use of the following Lemma, the proof of which is easily supplied.

LEMMA. *If $f(x)$, continuous in the interval $a \leqq x \leqq b$, vanishes at a and at b, and if $0 < \delta < b - a$, then there exists a value ξ such that*

$$\Delta f(\xi) = f(\xi + \delta) - f(\xi) = 0, \qquad a < \xi < b - \delta.$$

To prove the theorem we make use of formula (5). Take $d = x_{n+1}$, and form the corresponding functions $h_i(x)$. The functions $V_i(x)$ are positive in the interval $a \leqq x \leqq x_{n+1}$ if $0 < \delta \leqq (b - x_{n+1})/n$. Choose a fixed value of δ satisfying the relations (6). Since $V_1(x)$ is positive in $a \leqq x \leqq x_{n+1}$, it follows that the function $f(x)/V_1(x)$ is continuous in that interval, vanishing at the points x_i. We may consequently apply the Lemma to this function. Then

$$\Delta[f(\xi_{1i})/V_1(\xi_{1i})] = 0, \qquad x_i < \xi_{1i} < x_{i+1} - \delta, \qquad (i = 1, 2, \cdots, n).$$

The function $V_2(x)\, V_0(x+\delta)$ is positive in the interval $a \leqq x \leqq d$ so that the function

$$\frac{V_1(x+\delta)\, V_1(x)}{V_2(x)\, V_0(x+\delta)} \;\Delta\; \frac{f(x)}{V_1(x)}$$

is continuous in the interval $a \leqq x \leqq x_{n+1} - \delta$. Moreover the distance between two successive zeros ξ_{1i} of this function is greater than δ, so that the Lemma is applicable to this function. It follows that

$$\Delta \frac{V_1(x)\, V_1(x+\delta)}{V_2(x)\, V_0(x+\delta)} \;\Delta\; \frac{f(x)}{V_1(x)} \Bigg|_{x=\xi_{2i}} = 0, \quad \xi_{1i} < \xi_{2i} < \xi_{1\,i+1} - \delta,$$
$$(i = 1, 2, \cdots, n-1).$$

If we continue in this way, building up the formula (5) step by step, we see that

$$L^\delta f(\xi_{n1}) = 0, \qquad x_1 < \xi_{n1} < \xi_{n-1\,2} - \delta < x_{n+1} - n\delta.$$

The theorem is thus established.

COROLLARY. *If a function $f(x)$, continuous in $a \leqq x \leqq b$, assumes the same value as a function*

$$c_1u_1(x) + c_2u_2(x) + \cdots + c_nu_n(x)$$

at $(n+1)$ distinct points of $a \leqq x < b$, and if the set (1) *enjoys the property V in (a, b), then $L^\delta f(x)$ vanishes at an intermediate point for any positive value of δ sufficiently small.*

The proof is made by applying the theorem to the function

$$\bar{f}(x) = f(x) - c_1u_1(x) - c_2u_2(x) - \cdots - c_nu_n(x)\,,$$

noting that

$$L^\delta \bar{f}(x) \equiv L^\delta f(x).$$

4. *The Interpolatory Properties of the Functions $u_i(x)$.*

THEOREM II. *A necessary and sufficient condition that the vanishing of the function*

$$c_1u_1(x) + c_2u_2(x) + \cdots + c_nu_n(x)$$

at n distinct points of the interval $a \leqq x < b$ should imply the identical vanishing of this function is that the set (1) *possess the property V in (a, b).*

We begin with the sufficiency of the condition. Suppose then that the set (1) enjoys the property V in (a, b), and that a function

$$\phi(x) = c_1u_1(x) + c_2u_2(x) + \cdots + c_nu_n(x),$$

not identically zero, vanishes at n distinct points x_i:

$$a \leqq x_1 < x_2 < x_3 < \cdots < x_n < b.$$

Determine a constant d greater than x_n and less than b, and form the corresponding functions $h_i(x)$. Then

$$\phi(x) = a_1 h_1(x) + a_2 h_2(x) + \cdots + a_n h_n(x).$$

Suppose that

$$a_k \neq 0, \quad a_{k+1} = a_{k+2} = \cdots = a_n = 0, \quad k \leqq n.$$

By a remark in the introduction the property V holds for the functions $h_1(x), h_2(x), \cdots, h_{k-1}(x)$ in the interval (a, d). We may consequently apply the corollary of Theorem I to the difference operator $L^\delta{}_{k-1}$ defined by the relation

$$L^\delta{}_{k-1} f = V[h_1, h_2, \cdots, h_{k-1}, f].$$

Since $\phi(x)$ vanishes at n distinct points of (a, d), $a_k h_k(x)$ assumes the value of the function

$$-[a_1 h_1 + a_2 h_2 + \cdots + a_{k-1} h_{k-1}]$$

at these points. Since $n > k - 1$, it follows that

$$V[h_1, h_2, \cdots, h_{k-1}, a_k h_k], \quad \text{or} \quad a_k V_k(x)$$

vanishes at an interior point of (a, d) for any value of δ sufficiently small. This, however, is impossible since $V_k(x) > 0$ in (a, d). The sufficiency of the condition is thus established.

To prove the necessity of the condition we begin by assuming that no linear combination of the functions $u_i(x)$, which is not identically zero, vanishes at n distinct points of (a, b). It follows then that the determinant

$$|u_i(x_j)| = \begin{vmatrix} u_1(x_1) & u_1(x_2) & \cdots & u_1(x_n) \\ u_2(x_1) & u_2(x_2) & \cdots & u_2(x_n) \\ \cdot & \cdot & \cdot & \cdot \\ u_n(x_1) & u_n(x_2) & \cdots & u_n(x_n) \end{vmatrix}$$

is different from zero, no matter what positions the distinct points x_i assume in (a, b). Now choose an arbitrary constant d, $a < d < b$, and choose n distinct points x_i of the interval $b - (b - d)/n < x < b$. Determine linear combinations of the functions (1), $h_i(x)$, satisfying the conditions

$$h_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

This is possible since the determinant $|u_i(x_j)| \neq 0$. Now form the functions

$$V[h_1, h_2, \cdots, h_k], \qquad (k = 1, 2, \cdots, n).$$

We can show that all of these functions are different from zero in the interval $a \leqq x \leqq d$, if $\delta \leqq (b-d)/n$. For, suppose that $V[h_1, h_2, \cdots, h_k]$ vanished at a point x_0 of that interval. It would then be possible to determine a function

$$c_1h_1(x) + c_2h_2(x) + \cdots + c_kh_k(x)$$

not identically zero and vanishing at the points $x_0, x_0 + \delta, \cdots, x_0 + \overline{k-1}\,\delta$ of the interval $a \leqq x \leqq b - (b-d)/n$. This function would also vanish at the points

$$x_i \qquad (i = k+1, k+2, \cdots, n)$$

of the interval $b - (b-d)/n < x < b$. That is, the function would vanish at n distinct points of (a, b), contrary to assumption. If the functions $V_k(x)$ are not all positive, we have only to change the signs of certain of the functions $h_i(x)$. The proof is thus complete.

COROLLARY 1. *If the set* (1) *possesses the property V in* (a, b), *there exists a unique linear combination of the functions of the set taking on prescribed values at n arbitrary distinct points of* $a \leqq x < b$.

COROLLARY 2. *The property W implies the property V.*

For, Pólya showed that if the property W holds for the set (1), no linear combination of the functions of the set vanishes at n distinct (or coincident) points, unless it is identically zero.

We point out that it may be desirable to make use of the property V rather than the property W even though the functions (1) possess all the necessary derivatives. A case in point is that in which one wishes to distinguish between distinct and coincident zeros. For example, Theorem II shows that the function

$$c_1 + c_2x^3 \qquad (c_1^2 + c_2^2 \neq 0)$$

can not vanish at two distinct points of any interval. Yet the function x^3 has three coincident zeros at the origin.

THEOREM III. *If the set* (1) *possesses the property V in* (a, b), *it is possible to determine a linear combination of the functions of the set* changing sign *at k,* $[k < n]$, *distinct points of the interval* $a < x < b$. *Moreover, every linear combination which vanishes at* $n-1$ *points of that interval changes sign at each zero unless it is identically zero.*

Denote the k distinct points by x_i:

$$a < x_1 < x_2 < \cdots < x_k < b.$$

Choose $d > x_k$, and form the functions $h_i(x)$ corresponding. Consider the function

$$u(x) = F(x;\, x_1,\, x_2,\, \cdots,\, x_k) = \begin{vmatrix} h_1(x) & h_2(x) & \cdots & h_{k+1}(x) \\ h_1(x_1) & h_2(x_1) & \cdots & h_{k+1}(x_1) \\ \cdot & \cdot & \cdot & \cdot \\ h_1(x_k) & h_2(x_k) & \cdots & h_{k+1}(x_k) \end{vmatrix}.$$

This function is not identically zero, and vanishes at the points x_i. Since the property V holds for the set $h_1, h_2, \cdots, h_{k+1}$ in (a, d), u vanishes at no other points of (a, d). Furthermore, if we regard the x_i as variables, the function $F(x, x_1, x_2, \cdots, x_k)$ is different from zero if the variables remain distinct. F is a continuous function of its $(k+1)$ variables. Let the points $x, x_1, \cdots, x_k$ vary, always satisfying the inequalities

$$a < x < x_1 < x_2 < \cdots < x_k < d\ ,$$

until they coincide with the points

$$x_0,\ x_0 + \delta,\ x_0 + 2\delta,\ \cdots,\ x_0 + k\delta$$

respectively, all in the interval $a < x < d$. F then reduces to $V_k(x_0)$, which is positive. Hence $u(x)$ is positive if $x < x_1$. A similar proof shows that $u(x) < 0$ if $x_1 < x < x_2$. In this way we see that u changes sign at each of the points x_i.

If $k = n - 1$, the only linear combination of the given functions vanishing at the points x_i is the function $cF(x, x_1, x_2, \cdots, x_{n-1})$, the constant c being arbitrary. Every such function changes sign at the points x_i if $c \neq 0$, so that the theorem is completely established.

5. *The Remainder Formula.*

Theorem IV. *Let the set* (1) *possess the property V in (a, b); let $f(x)$ be continuous in $a \leqq x \leqq b$; and let $x_1, x_2, \cdots, x_{n-1}$ be arbitrary points of $a < x < b$. Then there exist functions $\phi(x)$ and $\psi(x)$ not identically zero satisfying the conditions*

$$\phi(x) = \sum_{i=1}^{n} c_i u_i(x), \qquad \phi(x_i) = f(x_i),$$
$$(i = 1, 2, \cdots, n-1)$$
$$\psi(x) = \sum_{i=1}^{n} a_i u_i(x), \qquad \psi(x_i) = 0.$$

If x_0 is any point of $a < x < b$, then

$$(7) \qquad f(x_0) = \phi(x_0) + \frac{L^\delta_{n-1} f(\xi)}{L^\delta_{n-1} \psi(\xi)} \psi(x_0) \qquad (a < \xi < b)$$

for any value of δ sufficiently small. Here

$$L^\delta_{n-1} f = V[h_1, h_2, \cdots, h_{n-1}, f], \qquad h_i = \sum_{j=1}^{n} c_{ij} u_j.$$

Choose a constant d greater than all the values x_i, $i = 0, 1, 2, \cdots, n-1$, and form the corresponding functions h_i. The property V holds for the set $h_1, h_2, \cdots, h_{n-1}$ in (a, d), and hence by Corollary 1 to Theorem II we may determine $\phi(x)$ as a linear combination of these functions:

$$\phi(x) = \sum_{i=1}^{n-1} k_i h_i(x), \qquad \psi(x) = \sum_{i=1}^{n} l_i h_i(x).$$

Now form the function

$$F(x) = f(x) - \phi(x) + C\psi(x),$$

where C is a constant to be determined. $F(x)$ vanishes at the $(n-1)$ points x_i, $i = 1, 2, \cdots, n-1$. C is to be determined so that $F(x_0) = 0$. This is possible since $\psi(x_0) \neq 0$. [Theorem II]. We now apply Theorem I to the function F, replacing the operator L^δ of that theorem by the operator L^δ_{n-1} defined by the equation

$$L^\delta_{n-1} f = V[h_1, h_2, \cdots, h_{n-1}, f].$$

We obtain in this way the following equation

$$L^\delta_{n-1} F(\xi) = 0 = L^\delta_{n-1} f(\xi) + C\, L^\delta_{n-1} \psi(\xi), \qquad a < \xi < b,$$

δ being any positive constant sufficiently small. This equation determines C since

$$L^\delta_{n-1} \psi(x) = l_n V_n(x) \neq 0.$$

$V_n > 0$ by hypothesis, and if l_n were zero, ψ would be a linear combination of $h_1, h_2, \cdots, h_{n-1}$, and would be identically zero since it vanishes at $n-1$ points of (a, d). We are thus led to formula (7) of the theorem.

6. *Applications.* As an example of formula (7) take

$$h_k(x) = u_k(x) = x^{k-1}, \qquad (k = 1, 2, 3, \cdots, n+1).$$

A function $\phi(x)$ satisfying the conditions

$$\phi(k\delta) = f(k\delta), \qquad (k = 0,\, 1,\, 2,\, \cdots,\, n-1)$$

is seen to be

$$f(0) + \frac{x}{\delta}\,\Delta^{\delta}{}_1(0) + \cdots + \frac{x(x-\delta)\cdots(x-n-2\delta)}{(n-1)!\,\delta^{n-1}}\Delta^{\delta}{}_{n-1}(0).$$

A function $\psi(x)$ vanishing at the points $0,\, \delta,\, 2\delta,\, \cdots,\, \overline{n-1}\,\delta$ is

$$x(x-\delta)(x-2\delta)\cdots(x-\overline{n-1}\,\delta).$$

In this case

$$L_n{}^{\delta_1}\psi = \Delta_n{}^{\delta_1}\psi = n!\;\delta_1{}^n.$$

Formula (7) gives us the relation

$$f(x) = f(0) + \cdots + \frac{x(x-\delta)\cdots(x-\overline{n-2}\,\delta)}{(n-1)!\,\delta^{n-1}}\,\Delta^{\delta}{}_{n-1}(0)$$

$$+ \frac{\Delta_n{}^{\delta_1}f(\xi)}{n!\;\delta_1{}^n}\; x(x-\delta)\cdots(x-\overline{n-1}\,\delta),$$

$$0 < \xi < (n-1)\delta, \qquad \xi < x.$$

This is Newton's interpolation formula with a remainder which was obtained by S. Bernstein.* The formula was used by him in order to affirm the analyticity of a real function not known to have derivatives.

In a similar way we might obtain for Lagrange's interpolation formula a remainder involving no derivatives.

As a further application we state without proof the following theorem.

THEOREM V.† *If the set* (1) *possesses the properpyt V in* (a, b), *and if the functions* $h_i(x)$ *corresponding to a value d in* $a < x < b$ *form an orthogonal set on* (a, d), *then the set* $h_i(x)$ *forms a set of oscillating functions; that is,* $h_i(x)$ *vanishes just* $(i-1)$ *times in* $a < x < d$, *and the zeros of* $h_i(x)$ *and of* $h_{i-1}(x)$ *occur alternately.*

Finally we state the following theorem, the proof of which will be given in a later article.

THEOREM VI.‡ *If the functions* $u_1(x),\, u_2(x),\, \cdots,\, u_n(x),\, f(x)$ *are*

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* "Sur la définition et les propriétés des fonctions analytiques d'une variable réele," *Mathematische Annalen*, Vol. 75 (1914), p. 452.

† Compare O. D. Kellogg, "The oscillations of functions of an orthogonal set," *American Journal of Mathematics*, Vol. 38 (1916), p. 1.

‡ Compare A. Haar, "Die Minkowskische Geometrie und die Annäherung an stetige Funktionen," *Mathematische Annalen*, Vol. 78 (1917-18), pp. 294-311.

*all continuous in the interval $a \leqq x \leqq b$, then a necessary and sufficient condition that there exist a unique function of approximation (in the sense of Tschebycheff *)*

$$\phi(x) = \sum_{i=1}^{n} a_i u_i(x)$$

to the function $f(x)$ in an arbitrary interval $a \leqq x \leqq d$ of (a, b) is that the set of functions $u_i(x)$ possess the property V in (a, b).

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* $\phi(x)$ is a function of approximation if and only if

$$\max | f(x) - \phi(x) | \leqq \max | f(x) - c_1 u_1(x) - c_2 u_2(x) - \cdots - c_n u_n(x) |$$

in (a, b), no matter what the constants c_i may be.

Note on Tschebycheff Approximation.

By D. V. Widder.

The problem of the representation of a continuous function $f(x)$ by Tschebycheff polynomials has been discussed in detail by various authors. The more general problem of the representation of $f(x)$ by a linear combination of prescribed continuous functions $u_1(x), u_2(x), \cdots, u_n(x)$, has received less consideration. J. W. Young * has discussed the general problem, and has imposed conditions on the functions $u_i(x)$ that guarantee the existence and uniqueness of the function of approximation (in the sense of Tschebycheff). These conditions are not necessary for the mere existence of the function of approximation. A necessary condition for the uniqueness was not discussed. The problem was also treated by A. Haar,† and it was found that a function of approximation always exists if the functions $u_i(x)$ are continuous. A necessary and sufficient condition for the uniqueness of the function was also obtained. The proofs given, however, are dependent on the theory of the geometry of Minkowski. Since the results are analytic, it seems desirable that a purely analytic proof be given. It is the purpose of the present note to give such a proof and to state the results in a new form.

In the author's paper on the interpolatory properties of a linear combination of continuous functions ‡ it was found that if the functions $u_i(x)$ possess a certain property, there designated as the property V, then every linear combination of them enjoys the more important interpolatory properties of a polynomial. We shall now show that this property gives a necessary and sufficient condition for the uniqueness of the function of approximation.

1. *The Existence of a Function of Approximation.*

Definition. If the functions

$$u_1(x),\ u_2(x),\ \cdots,\ u_n(x), \tag{1}$$

* "General theory of approximation by functions involving a given number of arbitrary parameters," *Transactions of the American Mathematical Society*, Vol. 8 (1907), pp. 331-344.

† "Die Minkowskische Geometrie und die Annäherung an stetige Funktionen," *Mathematische Annalen*, Vol. 78 (1917-18), p. 294.

‡ This number of the *American Journal of Mathematics*, p. 221.

and $f(x)$ are continuous in the interval $a \leqq x \leqq b$, then the function

$$\phi(x) = \sum_{i=1}^{n} a_i u_i(x)$$

is a function of approximation to $f(x)$ in (a, b) if and only if *

$$\max | f(x) - \phi(x) | \leqq \max | f(x) - c_1u_1(x) - c_2u_2(x) - \cdots - c_nu_n(x) |$$

for all real values of the constants c_i, the a_i being real constants.

THEOREM I. *There exists a function of approximation for every function $f(x)$ that is continuous in the interval $a \leqq x \leqq b$.*

Set

$$F(x) = \sum_{i=1}^{n} c_i u_i(x), \qquad y = f(x) - F(x).$$

Then the maximum of $| y |$ in $a \leqq x \leqq b$ depends in general on the values of c_i, and will be denoted by $m(c_1, c_2, \cdots, c_n)$. If $f(x)$ is a linear combination of the functions (1), $\phi(x)$ may be taken equal to $f(x)$, and the existence of the function of approximation is obvious. Otherwise $m(c_1, c_2, \cdots, c_n) > 0$ no matter what values the c_i assume. Now the function $m(c_1, c_2, \cdots, c_n)$ is continuous, since a small change in the c_i produces a small change in y and hence also in $\max | y |$.

Let M be the maximum of $| f(x) |$ in $a \leqq x \leqq b$, and for the present let us confine our attention to those functions $F(x)$ for which

$$| F(x) | \leqq 2M.$$

We may assume, without loss of generality, that the functions $u_i(x)$ are linearly independent. For, either they are all identically zero (in which case the theorem is true but trivial), or else there is some sub-set which is linearly independent. In the latter case the subsequent developments will apply for a smaller value of n. It follows then that the determinant

$$\Delta = | u_i(x_j) | \qquad (i, j = 1, 2, \cdots, n)$$

is not identically zero in the n variables x_j. Choose $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n$ so that $\Delta \neq 0$ for these values of the variables. It is then possible to determine functions

$$s_i(x) = \sum_{j=1}^{n} \alpha_{ij} u_j(x), \qquad (i = 1, 2, \cdots, n)$$

satisfying the conditions

* The notation $\max | F(x) |$ means the maximum absolute value of $F(x)$ in $a \leqq x \leqq b$.

$$s_i(\bar{x}_j) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \qquad (i, j = 1, 2, \cdots, n)$$

the constants α_{ij} being uniquely determined.

We can now express the arbitrary constants c_i on which $F(x)$ depends in terms of the values $F(\bar{x}_i)$ as follows:

$$\begin{aligned} F(x) &= c_1 u_1(x) + c_2 u_2(x) + \cdots + c_n u_n(x) \\ &= F(\bar{x}_1) s_1(x) + F(\bar{x}_2) s_2(x) + \cdots + F(\bar{x}_n) s_n(x) \\ &= F(\bar{x}_1) \sum_{j=1}^{n} \alpha_{1j} u_j(x) + F(\bar{x}_2) \sum_{j=1}^{n} \alpha_{2j} u_j(x) + \cdots + \\ &\qquad\qquad + F(\bar{x}_n) \sum_{j=1}^{n} \alpha_{nj} u_j(x) \\ &= u_1(x) \sum_{j=1}^{n} \alpha_{j1} F(\bar{x}_j) + u_2(x) \sum_{j=1}^{n} \alpha_{j2} F(\bar{x}_j) + \cdots + \\ &\qquad\qquad + u_n(x) \sum_{j=1}^{n} \alpha_{jn} F(\bar{x}_j). \end{aligned}$$

Hence

$$c_i = \sum_{j=1}^{n} \alpha_{ji} F(\bar{x}_j), \qquad (i = 1, 2, \cdots, n).$$

Since $| F(\bar{x}_j) | \leqq 2M$ it follows that

$$(2) \qquad | c_i | \leqq 2M \sum_{j=1}^{n} | \alpha_{ji} | = M_i, \qquad (i = 1, 2, \cdots, n).$$

That is, if we confine our attentions to functions $F(x)$ in absolute value not greater than $2M$, then the constants c_i are no longer arbitrary, but are restricted by the relations (2). The continuous function $m(c_1, c_2, \cdots, c_n)$ takes on a minimum value in the closed domain defined by these inequalities. If the minimum value is

$$m(a_1, a_2, \cdots, a_n) = \mu,$$

we shall show that the function

$$\phi(x) = \sum_{j=1}^{n} a_j u_j(x)$$

is a function of approximation. By the manner in which μ was obtained it is clear that

$$(3) \qquad \mu = \max | f(x) - \phi(x) | \leqq \max | f(x) - F(x) | = m(c_1, c_2, \cdots, c_n)$$

provided that $| F(x) | \leqq 2M$. In particular

$$\mu \leqq \max | f(x) | = m(0, 0, \cdots, 0) = M.$$

It remains to show that the relation (3) holds even if the c_i are not restricted by (2). In this case we may have

$$\max |F(x)| > 2M$$
$$\max |f(x) - F(x)| > 2M - M \geqq \mu.$$

Hence (3) holds in all cases, and $\phi(x)$ is in fact a function of approximation.

2. *Uniqueness of the Function of Approximation.*

THEOREM II. *If the functions $u_1(x), u_2(x), \cdots, u_n(x), f(x)$ are continuous in the interval $a \leqq x \leqq b$, then a necessary and sufficient condition that there exist a unique function of approximation*

$$\phi(x) = \sum_{i=1}^{n} a_i u_i(x)$$

*to the function $f(x)$ in the arbitrary interval $a \leqq x \leqq d$ of (a, b) is that the set of functions $u_i(x)$ possess the property V in (a, b).**

We begin with the sufficiency of the condition. Assuming the property V in (a, b) we wish to show the uniqueness of the function of approximation, $\phi(x)$, to $f(x)$ in the interval $a \leqq x \leqq d$. Here d is an arbitrary point in the open interval (a, b). Since the function

$$y = f(x) - \phi(x) \tag{4}$$

is continuous, $|y|$ takes on its maximum, m, at least once in $a \leqq x \leqq d$. Consider the set of points (x, y) on the curve defined by (4) for which $y = \pm m$. We shall say that the set presents a change of sign between (x', y') and (x'', y'') if $y' = -y'' = \pm m$, and if there exists no point (x, y) of the set for which $x' < x < x''$. It will now be shown that the set of points presents at least n changes of sign. For, suppose that there were only k changes of sign $[k < n]$. If the set has a change of sign between (x', y') and (x'', y''), choose a value x_i such that $x' < x_i < x''$. We thus obtain k values

$$x_1 < x_2 < \cdots < x_k.$$

It was shown in the author's paper already cited (Theorem III) that if the set (1) possesses the property V in (a, b), then it is possible to determine a linear combination of the functions of the set,

$$H(x) = \sum_{i=1}^{n} c_i u_i(x),$$

changing signs at each of the distinct points x_i. If $H(x)$ has further zeros they will all be greater than x_k.

The points x_i divide the interval (a, d) into $k + 1$ sub-intervals such that y can not take on the values m and $-m$ in the same interval and takes

* For the precise definition of the property V, see the author's paper already cited.

on both values in two adjoining intervals. We can now determine a constant η such that the function

$$y - \eta H(x)$$

is less than m in absolute value in (a, d). Suppose for definiteness that $y = +m$ at some point of (a, x_1). Then the following relations hold:

$$(5) \qquad \begin{array}{rcll} -m+\epsilon \leqq y \leqq m & & \text{in} & a \leqq x \leqq x_1 \\ -m \leqq y \leqq m-\epsilon & & \text{in} & x_1 \leqq x \leqq x_2 \\ -m+\epsilon \leqq y \leqq m & & \text{in} & x_2 \leqq x \leqq x_3 \\ \cdots\cdots & & & \cdots\cdots \\ \cdots\cdots & & & \cdots\cdots \end{array}$$

Here ϵ is some positive constant sufficiently small. Determine η so that $\eta H(x) > 0$ in (a, x_1), and such that

$$|\eta H(x)| < \epsilon \qquad \text{in} \qquad a \leqq x \leqq b.$$

Then

$$(6) \qquad \begin{array}{rcll} -\epsilon < -\eta H(x) < 0 & & \text{in} & a < x < x_1 \\ 0 < -\eta H(x) < \epsilon & & \text{in} & x_1 < x < x_2 \\ -\epsilon < -\eta H(x) < 0 & & \text{in} & x_2 < x < x_3 \\ \cdots\cdots & & & \cdots\cdots \\ \cdots\cdots & & & \cdots\cdots \end{array}$$

Combining the relations (5) with the inequalities (6) we see that

$$-m < y - \eta H(x) < m \qquad \text{in} \qquad a \leqq x \leqq b.$$

Hence $\phi(x) + \eta H(x)$ is a linear combination of the functions (1) which has closer approximation to $f(x)$ than $\phi(x)$. This contradicts the hypothesis that $\phi(x)$ is a function of approximation.

It is now a simple matter to show that $\phi(x)$ is unique. For, suppose that there were another function of approximation $\bar{\phi}(x)$. Form the function

$$\psi(x) = [f(x) - \phi(x)] - [f(x) - \bar{\phi}(x)].$$

When $y = m$, $\psi(x) \geqq 0$ since $f(x) - \bar{\phi}(x)$ is not greater than m; and when $y = -m$, $\psi(x) \leqq 0$. Since the set of points (x, y) presents at least n changes of sign, it follows that $\psi(x)$ vanishes at least n times in $a \leqq x \leqq d$. This is impossible unless $\psi(x)$ vanishes identically, since it is a linear combination of the functions (1)*. The contradiction shows that $\phi(x)$ is unique.

We turn now to the necessity of the condition. We shall show that if the function of approximation to $f(x)$ in the arbitrary interval $a \leqq x \leqq d < b$ is unique, then no linear combination of the functions (1) vanishes n times in

* See Theorem II of the author's paper already cited.

$a \leqq x \leqq d$, unless it is identically zero, and hence that the property V holds in (a, b).* For, suppose that the function

$$h(x) = \sum_{i=1}^{n} b_i u_i(x),$$

not identically zero, vanishes at n distinct points x_i, $(i = 1, 2, \cdots, n)$ of $a \leqq x \leqq d$. Then the determinant

$$| u_i(x_j) | \qquad (i, j = 1, 2, \cdots, n)$$

vanishes. We may consequently determine constants $a_1, a_2, \cdots, a_n$, not all zero, such that

$$\sum_{i=1}^{n} a_i u_j(x_i) = 0, \qquad (j = 1, 2, \cdots, n)$$

Then if

$$F(x) = \sum_{i=1}^{n} c_i u_i(x) ,$$

it follows that

$$\sum_{i=1}^{n} a_i F(x_i) = 0 \tag{7}$$

no matter what value the constants c_i assume. We can now show that for a suitable choice of the functon $f(x)$ two functions of approximation exist. Choose $f(x)$ such that

$$\begin{aligned} &f(x_i) = a/| a_i | && \text{if} && a_i \neq 0, \\ &f(x_i) = 0 && \text{if} && a_i = 0, \\ &| f(x) | \leqq 1. \end{aligned}$$

Then

$$\max | f(x) - F(x) | \geqq 1.$$

For if this were not the case, $F(x)$ would have the same sign as $f(x)$ at each of the points x_i [for which $f(x_i) \neq 0$], and hence we should have

$$\sum_{i=1}^{n} a_i F(x_i) > 0.$$

But this contradicts the equation (7). Hence any function $F(x)$ such that $| f(x) - F(x) | \leqq 1$ is a function of approximation. In particular $F(x) \equiv 0$ is a function of approximation since $| f(x) | \leqq 1$. We now restrict $f(x)$ further so that

$$| f(x) - h(x) | \leqq 1.$$

This restriction is compatible with the foregoing restrictions since $h(x)$ vanishes at the points x_i. Then $h(x)$ is also a function of approximation, not identically zero, and we have obtained a contradiction. The theorem is thus completely established.

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* See Theorem II of the author's paper already cited.

Generalizations of Waring's Theorem on Fourth, Sixth, and Eighth Powers.

By L. E. Dickson.

In a recent number of this Journal, I investigated positive quadratic forms which represent every positive integer p. The fact that there are so many such forms led me to conjecture the existence of numerous forms of every degree n which represent every p. The form $a_1x_1^n + \cdots + a_mx_m^n$ is said to be of order m and *weight* $a_1 + \cdots + a_m$. I soon found * all such forms M of minimum weight when $n = 3, 4, 5$, while the cases $n = 6, 7$ have been treated in Chicago theses. Most of these forms M have the important property that their order is considerably smaller than their weight, while order and weight are equal for $x_1^n + \cdots + x_m^n$. The new generalizations of Waring's theorem on the latter form therefore involve a material reduction in order.

The theorems for $n = 4, 6, 8$ proved here have the like improvement of a reduction in order over the known results for Waring's theorem for the same values of n.

For $s = 0, 1, \cdots, 10$, I prove that every p is a sum of s doubles of biquadrates and $37 - 2s$ biquadrates, and hence p is represented by a form of order $37 - s$. The case $s = 0$ gives the best known complete result for Waring's theorem on biquadrates.* Hence there is a reduction of the order by s, whose maximum is 10.

Fleck showed that 2451 sixth powers suffice. Kempner gave a reduction to 970, and Baer to 478. My Theorems 2, 3, 4 give reductions to forms of orders 223, 277, 190, respectively.

Hurwitz proved that 36119 eighth powers suffice. By a simple remark, Kempner reduced this to 31353. My final theorem gives a reduction to a form of order 2690.

* Recent numbers of the *Bulletin of the American Mathematical Society, American Mathematical Monthly, Annals of Mathematics.*

† Liouville proved that 53 biquadratets suffice. Reductions were made in turn by Réalis, Lucas, Fleck, Landau, and Wieferich (to 37). See the writer's *History of the Theory of Numbers*, Vol. 2, pp. 717-20. In his thesis, *Beiträge zum Waringschen Problem*, Göttingen, 1913, Baer modified the Landau-Wieferich type of proof and showed that 34 biquadrates suffice for $48k + 1$ and $48k + 33$. Some computations made in that proof are avoided in the present proof (Lemma 8).

PART I. LEMMAS ON nTH POWERS.

LEMMA 1. *If s and t are positive, n is a positive integer, and if $s \geqq g + t$, there exists a positive integer i such that*

$$(1) \qquad g \leqq s - ti^n < g + tR, \quad R = r^n - (r-1)^n, \quad r = [s-g)/t]^{1/n}.$$

With the understanding that r is the positive real root, we have $r \geqq 1$. Write $r = i + f$, where $0 \leqq f < 1$, and i is an integer $\geqq 1$. Since $i \leqq r$,

$$g = s - tr^n \leqq s - ti^n,$$

as desired in (1). Next,

$$s - ti^n - g = tw, \qquad w = r^n - (r-f)^n,$$
$$R - w = (r-f)^n - (r-1)^n > 0,$$

since $r - f$ exceeds $r - 1$, which is $\geqq 0$. This proves Lemma 1. To prove that

$$(2) \qquad R < n\, r^{n-1},$$

write $\rho = r - 1$. Then

$$R = (r - \rho)Q = Q = \sum_{i=0}^{n-1} r^{n-1-i} \rho^i.$$

Since $\rho < r$, each of the n terms of Q is $\leqq r^{n-1}$. When $g \geqq 0$, then $r^n \leqq s/t$ and we have

LEMMA 2. *If s and t are positive, n is a positive integer, $g \geqq 0$, and $s \geqq g + t$, there exists a positive integer i such that*

$$(3) \qquad g \leqq s - t\, i^n < g + n(ts^{n-1})^{1/n}.$$

If $s < t$, then $s^n < t\, s^{n-1}$, and s is less than the radical and hence less than its product by n. Then (3) holds with $g = 0$, $i = 0$. Thus Lemma 2 implies

LEMMA 3. *If s and t are positive and n is a positive integer, there exists an integer $i \geqq 0$ such that*

$$(4) \qquad 0 \leqq s - t\, i^n < n(ts^{n-1})^{1/n}.$$

Hence there exists an integer $i_1 \geqq 0$ such that

$$0 \leqq s_1 = s - t\, i_1{}^n < m\, s^\nu, \qquad m = n\, t^{1/n}, \quad \nu = (n-1)/n.$$

Applying this to s_1 in place of s, we see that there exists an integer $i_2 \geqq 0$ such that

$$0 \leqq s_2 - s_1 - t i_2^n < m s_1^{\nu} < m^{1+\nu} s^{\nu^2}.$$

By induction on r, we obtain

LEMMA 4. *If s and t are positive and n is a positive integer, there exist integers $i_1, \cdots, i_r$, each $\geqq 0$, such that*

$$s = t i_1^n + \cdots + t i_r^n + s_r, \qquad \nu = (n-1)/n, \tag{5}$$

$$0 \leqq s_r < (n t^{1/n})^e s^{\nu^r}, \qquad e = 1 + \nu + \nu^2 + \cdots + \nu^{r-1}. \tag{6}$$

PART II. FOURTH POWERS.

We write $[b, d]$ for a sum of b biquadrates and d doubles of biquadrates. Our goal is

THEOREM 1. *Every positive integer is represented by* $[17, 10]$.

The main part of the proof applies to all integers exceeding a limit just under 95^4. We first prove

LEMMA 5. *Every positive integer $s \leqq 95^4$ is represented by* $[17, 10]$.

By Lemma 4 with $t = 2$, $n = 4$, $r = 5$, we have

$$s = 2i_1^4 + \cdots + 2i_5^4 + s_5, \qquad 0 \leqq s_5 < cd, \tag{7}$$
$$c = 2^{9e/4}, \qquad d = 95^f, \qquad f = 3^5/4^4, \qquad e = 3.050781.$$

But $d < 78$ since $95^{243} < 78^{256}$, and $c < 2^7$. Hence $s_5 < 9984$. We next prove * that every positive integer $p \leqq 9986$ is represented by $[17, 5]$. If $p \geqq 2 \cdot 7^4 = 4802$, $p = 2 \cdot 7^4 + q$, $0 \leqq q \leqq 5184 = 4 \cdot 6^4$. Subtracting $2 \cdot 6^4$ from q when possible, we see that it remains to prove that every positive integer $\leqq 2 \cdot 6^4 = 2592$ is represented by $[17, 3]$. It is either $\leqq 2 \cdot 5^4 = 1250$ or exceeds the latter by an integer $u \leqq 1342$. If $u \geqq 1250$, $u \leqq 2 \cdot 5^4 + 92$. If $2 \cdot 4^4 = 512 \leqq u < 1250$, $u \leqq 2 \cdot 4^4 + 738$. Hence it remains only to show that every integer $\leqq 738$ is represented by $[17, 1]$. While a direct proof is not tedious, it has been verified in several ways.†

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* Hence Lemma 5 holds when 95 is replaced by 108.756, and therefore for $s < 1.399 \times 10^8$. See Lemmas 10, 11.

† Since [17, 1] is a partition of all the forms in *Bulletin of the American Mathematical Society*, Vol. 33 (1927), pp. 319-27, in particular of that in § 8.

LEMMA 6. *If p is odd, $6p^2$ is represented by* [3, 4]. *For any positive integer k, $6k^2$ is represented by* [6, 3].

It is known that every positive integer p, which is not of the form $q = 4^r(16s + 14)$ is represented by $a^2 + b^2 + 2c^2$. The identity

$$(8) \qquad 6(a^2 + b^2 + 2c^2)^2 \equiv (a + b)^4 + (a - b)^4 + (2c^4) + 2(a + c)^4 + 2(a - c)^4 + 2(b + c)^4 + 2(b - c)^4$$

shows that $6p^2$ is represented by [3, 4]. Next, $16s + 14$ is congruent to 6 modulo 8 and is known to be a sum of three squares. If we multiply all their roots by 2^r, we see that q is a sum of three squares. The identity

$$(9) \qquad 6(a^2 + b^2 + c^2)^2 \equiv (a + b)^4 + (a - b)^4 + (a + c)^4 + (a - c)^4 + (b + c)^4 + (b - c)^4 + 2a^4 + 2b^4 + 2c^4$$

shows that $6q^2$ is represented by [6, 3]. Since any positive integer k is either a p or a q, and since representation by [3, 4] implies that by [5, 3] and hence by [6, 3], with one biquadrate zero, Lemma 6 is proved.

LEMMA 7. *Every positive integer $24k + 6$ is represented by* [15, 10]. *Every $24k + 12$ is represented by* [12, 11].

For, every positive integer $4k + 1$ or $4k + 2$ is a sum of three squares, exactly 1 or 2 of which are odd, respectively. We apply Lemma 6. Since $6(4k + 2) = 6r^2 + 6s^2 + 6p^2$, where r and s are odd, it is represented by the sum [12, 11] of [3, 4], [3, 4], and [6, 3]. Similarly, $6(4k + 1)$ is represented by the sum [15, 10] of [3, 4], [6, 3], [6, 3].

LEMMA 8. *Every positive integer $48k + 1$ or $48k + 33$ is represented by* [10, 12] *if it exceeds $r^4 - 480$, where $r = 95$ or 93, respectively.*

For $a = 12n \pm 1$, $a^4 - 1$ is the product of

$$a \mp 1 = 12n, \quad a \pm 1 = 2(6n \pm 1), \quad a^2 + 1 = 2(72n^2 \pm 12n + 1).$$

Hence

$$1 - a^4 = -48q, \quad q = n(\pm 144n^2 + 18n \pm 1) \equiv n(2n \pm 1) \pmod{16}.$$

We readily prove that q ranges with n over a complete set of residues modulo 16. For, if $x(2x \pm 1) \equiv n(2n \pm 1) \pmod{16}$, evidently $x \equiv n \pmod 2$, whence $2x^2 \equiv 2n^2 \pmod 8$. Then the initial congruence gives $x \equiv n \pmod 8$ and therefore also $x \equiv n \pmod{16}$.

Since $n = 16 - m$ implies $n(2n + 1) \equiv m(2m - 1) \pmod{16}$, the

values of $n(2n+1)$ for $n=0, 1, \cdots, 7$, and the values of $n(2n-1)$ for $n=1, \cdots, 8$, together form a complete set of residues modulo 16. The corresponding a's have the maximum $12 \cdot 8 - 1 = 95$.

Hence there exists a positive integer $a \leqq 95$ such that $1 - a^4 \equiv 48\alpha$ (mod $48 \cdot 16$), where α is any prescribed residue modulo 16. Take $\alpha = 6 - k$, where $s = 48k + 1$ is the first integer in Lemma 8. Then there is an integer h such that

$$1 - a^4 = 48(6-k) + 48 \cdot 16\, h, \quad s - a^4 = 48(6 + 16\, h) = 2^4 \cdot 6\,(8\, h + 3).$$

Since $s > 95^4 - 480$, we have $h > -1$. Thus $h \geqq 0$, and $8h + 3$ is a sum of three odd squares. Lemma 6 shows that $6(8h+3)$ is represented by $[9, 12]$. Since the same is true of its product $s - a^4$ by 2^4, s is represented by $[10, 12]$.

Second, let $s = 48k + 33$. For $a = 3t$, $t = 4m \pm 1$,

$$33 - a^4 = 48Q, \quad Q = -1 - 27\,(16m^4 \pm 16m^3 + 6m^2 \pm m),$$
$$Q \equiv (7m \mp 1)(2m \pm 1) \equiv 7n(2n \mp 1) \quad (\text{mod } 16),$$

where $n = m \mp 7$.

As in the first case, the values of $n(2n+1)$ for $n = 8, \cdots, 15$ and of $n(2n-1)$ for $n = 0, -1, \cdots, -7$ together form a complete set of residues modulo 16. For the former, $m = n - 7 = 1, \cdots, 8$; $t = 4m - 1 = 3, 7, \cdots, 31$. For the latter, $m = n + 7 = 0, 1, \cdots, 7$; $t = 4m + 1 = 1, 5, \cdots, 29$. Hence * when t ranges over the 16 positive odd integers $\leqq 31$, Q ranges over a complete set of residues modulo 16. The proof now proceeds as in the first case.

LEMMA 9. *Every positive integer $s = 48k + 9$ or $48k + 25$ is represented by $[13, 11]$ if it exceeds $r^4 - 456$, where $r = 93$ or 95, respectively.*

Since $s - 24 \equiv 33$ or 1 (mod 48), the proof of Lemma 8 shows the existence of a positive integer a such that

$$s - 24 - a^4 = 2^4 \cdot 6(8h+3), \quad s - a^4 = 24(16l + 13),$$

where h is an integer $\geqq 0$, and $l = 2h$. Since $16l + 13$ is $\equiv 1$ (mod 4), it is a sum of an odd and two even squares. The odd one is $\equiv 1$ (mod 8) and hence is $\equiv 1$ or 9 (mod 16). An even square is $\equiv 0$ or 4 (mod 16). Hence

* A second proof follows from the fact that $t^4 \equiv \tau^4$, $\tau \equiv tx$ (mod 16^2) imply $x^4 \equiv 1$ (mod 16^2), whence $x = \pm 1$ (mod 64).

any sum of two even squares is $\equiv 0$, 4, or 8. One of the latter increased by 1 or 9 shall give 13 (mod 16), whence the summands are 4 and 9. Hence

$$16l + 3 = u^2 + v^2 + w^2, \quad u^2 \equiv 9, \quad v^2 \equiv 4, \quad w^2 \equiv 0 \pmod{16}.$$

We may take u, v, w all $\geqq 0$. Since $u = 4n \pm 1$, $9 \equiv \pm 8n + 1 \pmod{16}$ and n is odd. Thus $u \equiv 3$ or $5 \pmod 8$. Next, v is the double of an odd integer. Hence both u and v are sums of three squares. The identity

$$(10) \quad 24(a^2 + b^2 + c^2)^2 \equiv 2(a + b + c)^4 + 2(a + b - c)^4 + 2(a - b + c)^4 + 2(a - b - c)^4 + (2a)^4 + (2b)^4 + (2c)^4$$

shows that $24u^2$ and $24v^2$ are both represented by $[3, 4]$. Since $w = 2m$, $24w^2 = 2^4 \cdot 6m^2$ is represented by $[6, 3]$ by Lemma 6. Hence $s - a^4$ is represented by

$$[3, 4] + [3, 4] + [6, 3] = [12, 11].$$

Proof of Theorem 1. Any integer $24n + 1$ is of one of the forms $48k + 1$, $48k + 25$. Hence Lemmas 8 and 9 together state that every positive integer $24n + 1$ or $24n + 9$, exceeding $95^4 - 456$, is represented by $[13, 11]$ and hence by $[15, 10]$. By Lemma 7, every positive $24n + 6$ is represented by $[15, 10]$, and every $24n + 12$ by $[14, 10]$.

By adding 1^4 or $1^4 + 1^4$ to these types $24n + 1$, 6, 9, 12, we get $24n + 2$ or 3, $24n + 7$ or 8, $24n + 10$ or 11, $24n + 13$ or 14. Next,

$$\begin{aligned}
24n &= \{24(n - 7) + 6\} + 3^4 + 3^4, \\
24n + 4 &= \{24(n - 1) + 12\} + 2^4, \\
24n + 15 &= \{24(n - 3) + 6\} + 3^4, \\
24n + 17 &= \{24n + 1\} + 2^4, \\
24n + 18 &= \{24(n - 3) + 9\} + 3^4, \\
24n + 20 &= \{24(n - 1) + 12\} + 2^4 + 2^4, \\
24n + 21 &= \{24(n - 3) + 12\} + 3^4.
\end{aligned}$$

The numbers involved are all positive if $n \geqq 7$. Those with $n < 7$ are $\leqq 24 \cdot 6 + 23 = 137$. By adding 1^4 to $24n + 4$, 15, 18, 21, we get $24n + 5$, 16, 19, 22. Hence the latter may be obtained by adding just two biquadrates to numbers of the types $24f + 6$, 9, 12. We have now reached all the types $24n + 0, 1, \cdots, 22$. Every such number N exceeding $95^4 - 294$ is represented by $[17, 10]$. For it was obtained by adding at most two biquadrates with the maximum sum $3^4 + 3^4$ to a number $p = 24k + 1$, 6, 9, 12. Since $N \leqq p + 162$, p exceeds $95^4 - 456$ and hence is represented by $[15, 10]$.

Hence N is represented by $[17, 10]$. Finally, $j = 24n + 23$ is obtained by adding 2 to $24n + 21$ and hence by adding $3^4 + 1^4 + 1^4$ to $24(n-3) + 12$. Thus j is represented by $[3, 0] + [14, 10] = [17, 10]$.

We have now proved that every integer exceeding $95^4 - 294$ is represented by $[17, 10]$. The same is true of those $\leqq 95^4$ by Lemma 5. Hence all are represented, as stated in Theorem 1.

The form $[17, k]$. Since this probably represents all positive integers when $k \geqq 1$, it is desirable to find the largest integer L_k for which we can prove by Lemmas 1-3 that every integer $< L_k$ is represented by $[17, k]$.

LEMMA 10. *Every positive integer* $p < 7848$ *is represented by* $[17, 1]$.

This is known for $p \leqq 4100$ (*Bulletin, l. c.*, § 8). We apply Lemma 1 with $n = 4$, $t = 2$, $g = 1249$, and write $\rho = 2r - 1$. Hence if $s \geqq 1251$, there exists a positive integer i such that

$$(11) \qquad 1249 \leqq \sigma < 1249 + \rho^3 + \rho, \quad \sigma = s - 2i^4, \quad 2r^4 = s - 1249.$$

The table of Bretschneider * shows that 1248 is the last integer $\leqq 4100$ which is not a sum of 17 biquadrates. Hence σ will be such a sum if the upper limit in (11) is 4101. Then

$$\rho^3 + \rho = 2852, \quad \rho = 14.1578, \quad r^4 = 3299.32, \quad s = 7847.6.$$

LEMMA 11. *Every positive integer* $< L_k$ *is represented by* $[17, k]$, *where* $L_2 = 20226$, $L_3 = 74425$, $L_4 = 421450$, $L_5 = 4148791$, $L_6 = 85388000$.

Taking the upper limit in (11) to be 7848, $L_2, L_3, \cdots$, we get

$\rho^3 + \rho$	ρ	r	r^4	s
6599	18.7391	9.8695	9488.13	20225.26
18977	26.66075	13.83038	36587.8	74424.6
73176	41.8190	21.4095	210100.1	421449.2
420201	74.896227	37.94811	2073771	4148791
4147542	160.66623	80.83312	42693500	85388000

PART III. SIXTH POWERS.

We write $(1_r, 8_s)$ for a sum of r sixth powers and the products of s sixth powers by 8.

THEOREM 2. *Every positive integer is represented by* $f = (1_{115}, 8_{108})$ *in sixth powers.*

* *Journal für Mathematik*, Vol. 46 (1853), pp. 1-23.

We employ the identity due to A. J. Kempner: *

$$(12)\qquad 120(a^2+b^2+c^2+d^2)^3 \\ = \sum_8 (a \pm b \pm c \pm d)^6 + \sum_4 (2a)^6 + 8 \sum_{12} (a \mp b)^6.$$

Hence the product of 120 by the cube of any integer $\geqq 0$ is represented by $(1_{12}, 8_{12})$. Any positive integer p is a sum of nine cubes. Hence $120p$ is represented by $(1_k, 8_k)$, $k = 9 \cdot 12$. It is readily shown that any integer is congruent to a sum of 7 sixth powers with respect to 3, 5, or 8 as modulus, and hence with respect to their product 120 as modulus. Hence if A is an integer $\geqq 7 \cdot 119^6$, $A = \sigma + 120p$, where $p \geqq 0$ and σ is a sum of 7 sixth powers. Hence A is represented by f.

It remains to prove this also when $A < 7 \cdot 119^6$. Note that $7 < 2^3$, $119 < 128 = 2^7$. We shall prove that every positive integer $s \leqq 2^{45}$ is represented by $(1_{43}, 8_{16})$ and hence by f.

We apply Lemma 4 with $t = 1$, $n = 6$, $r = 15$, whence $\nu = 5/6$. The inequalities (6) still hold if we increase s to 2^{45} and increase e to 6 by extending the geometrical progression to infinity. Thus

$$0 \leqq s_{15} < 6^6\, 2^m, \quad m = 45(5/6)^{15} < 3,$$

since

$$15 \log 1.2 = 1.187718 > 1.176091 = \log 15, \quad (6/5)^{15} > 15.$$

But $6^6\, 2^3 < 9^6$ since $2^3 < 3^2$. Hence by (5), s is a sum of 15 sixth powers $s \geqq 0$ and an integer m, where $0 \leqq m < 9^6$. Then $m = 3^6\alpha + \beta$, $0 \leqq \alpha \leqq 3^6$. $0 \leqq \beta \leqq 3^6$. We shall show that α and β are each represented by $(1_{14}, 8_8)$, whence m is represented by $(1_{28}, 8_{16})$. For, $3^6 < 12 \cdot 64$. When u, v, w are chosen from $0, 1, \cdots, 7$, $u + 8v + 2^6 w$ is represented by $(1_7, 8_7, 1_7)$. When $u, v = 0, 1, \cdots, 7$ and $z = 1, 2, 3$, $u + 8v + 2^6(8 + z)$ is represented by $(1_7, 8_7, 8_1, 1_3)$.

THEOREM 3. *Every positive integer is represented by* $(1_{178}, 8_{99})$ *in sixth powers.*

In the proof of Lemma 6 it was shown that every positive integer k is represented by at least one of $a^2 + b^2 + 2c^2$ and $a^2 + b^2 + c^2$. In the respective cases we apply (12) with $d = c$ or $d = 0$ and see that one term vanishes, whence $120k^3$ is represented by $(1_{12}, 8_{11})$ or $(1_{11}, 8_{12})$, and hence always by $(1_{19}, 8_{11})$. The rest of the proof of Theorem 2 applies here.

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* *Über das Waringsche Problem*, Dissertation, Göttingen, 1912, p. 47. Extract in *Mathematische Annalen*, Vol. 72 (1912), p. 396.

THEOREM 4. *Every positive integer S is represented by $g = (1_{10}, 2_{90}, 3_{72}, 4_{18})$ in sixth powers.*

We employ facts proved by Baer, *l. c.*, pp. 41-51. For at least one of the values 273 and 281 of σ, $S - \sigma$ is congruent to a sum $\sum_7$ of 7 sixth powers modulo 360, each $< 360^6 < 2^{9 \cdot 6}$. Assume first that $S \geqq 2^{968}$. Write $S = \sum_7 + l$. Then $l > 2^{968} - 7 \cdot 2^{54} > 2^{968} - 2^{57} > 2^{967}$, and $l \equiv \sigma \pmod{360}$. By an intricate proof, Baer showed that every such l is a sum of three sixth powers together with the product by 1440 of the sum of the cubes of six doubles of positive odd integers. Each such double is a sum of three squares. The identity

$$1440\,(a^2 + b^2 + c^2)^3$$

$$\equiv 2 \sum_{12} (2a \pm b \pm c)^6 + 2 \sum_{3} (2a)^6 + 3 \sum_{12} (2a \pm b)^6 + 4 \sum_{3} a^6$$

employed six times shows that l is represented by $(1_3, 2_{96}, 3_{72}, 4_{18})$. Hence $S = \sum_7 + l$ is represented by g.

We next prove that S is represted by g if $0 \leqq S \leqq 2^{968}$. Write $S = k^6 + 2s$, where $k = 0$ or 1 according as S is even or odd. Apply Lemma 4 with $t = 1$, $n = 6$, $r = 40$. The inequalities (6) still hold if we increase s to 2^{967} and increase e to 6 by extending the geometrical progression to infinity. Thus

$$s = \sum_{j=1}^{40} i_j^{\,6} + s_{40}, \quad 0 \leqq s_{40} < 6^6\, 2^m, \quad m = 967(5/6)^{40}.$$

But

$$625 = 5^4 < 3^4\, 2^3 = 648, \quad (5/6)^4 < \tfrac{1}{2}, \quad 2^{10} = 1024 > 967, \quad m < 1;$$

$$\sigma = 2s_{40} < 4 \cdot 6^6, \quad \sigma = 3^7\alpha + \beta, \quad 0 \leqq \beta < 3^7, \quad \alpha < \frac{4 \cdot 6^6}{3^7} = \frac{4 \cdot 2^6}{3} < 86.$$

Now α is a sum of 63 sixth powers, which are all 0 or 1 if $\alpha < 64$, while one is 2^6 if $\alpha \geqq 64$. Hence $3^6 \cdot 3\alpha$ is a sum of 63 triples of sixth powers. Next, $\beta = p \cdot 3^6 + \delta$, $p = 0$, 1 or 2, $0 \leqq \delta < 3^6$. Also, $\delta = 3 \cdot 2^6 q + \epsilon$, $q = 0$, 1, 2, or 3, $0 \leqq \epsilon < 3 \cdot 2^6$. Evidently $\epsilon = \lambda$, $2^6 + \lambda$ or $2 \cdot 2^6 + \lambda$, where $0 \leqq \lambda \leqq 63$. An even λ is $2u$, where $u \leqq 31$, and is a sum of u doubles of sixth powers, all unity. An odd λ is $1 + 2u$. Hence every λ, ϵ, δ, β, σ is represented by respectively $(1, 2_{31})$, $(1_3, 2_{31})$, $(1_3, 2_{31}, 3_3)$, $(1_5, 2_{31}, 3_3)$, $(1_5, 2_{31}, 3_{66})$. Hence every $S \leqq 2^{968}$ is represented by $(1_6, 2_{71}, 3_{66})$ and therefore by f.

PART IV. EIGHTH POWERS.

A. Hurwitz * gave the identity

$$5040\,(a^2+b^2+c^2+d^2)^4 \equiv \sum_{48}(2a\pm b\pm c)^8 + 6\sum_{4}(2a)^8$$
$$+\,6\sum_{8}(a\pm b\pm c\pm d)^8 + 60\sum_{12}(a\pm b)^8.$$

Any positive integer is of the form $5040\,Q+R$, where $0\leqq R\leqq 5039$. Since Q is a sum of the fourth powers of 37 integers (Part II), each a sum of four squares, $5040\,Q$ is represented by 37 sums of $(1_{48}, 6_{12}, 60_{12})$. Each R is of the form 2^8u+v, $0\leqq u\leqq 19$, $0\leqq v\leqq 225=2^8-1$. Hence R is represented by 1_{274}.

We obtain a better result as follows: †

$$\begin{array}{lll} R=6\cdot 2^8e+f, & e=0,1,2,3; & 0\leqq f<6\cdot 2^8; \\ f=2^8g+h, & 0\leqq g\leqq 5, & 0\leqq h<2^8; \\ h=60j+k, & 0\leqq j\leqq 4, & 0\leqq k<60; \\ k=6l+m, & 0\leqq l\leqq 9, & 0\leqq m\leqq 5. \end{array}$$

Hence k, h, f, R are represented by respectively

$$(1_5, 6_9),\quad (1_5, 6_9, 60_4),\quad (1_{10}, 6_9, 60_4),\quad (1_{10}, 6_{12}, 60_4),$$

the last being of order 26. This proves

THEOREM 5. *Every positive integer is represented by* $(1_{1786}, 6_{456}, 60_{448})$ *of order* 2690 *in eighth powers.*

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* *Mathematische Annalen*, Vol. 65 (1908), pp. 424-7.

† Avoiding 60, we may use $h=6n+p$, $n\leqq 42$, $p\leqq 5$. Hence R is represented by $(1_{10}, 6_{54})$, of order 55.

On Complete Systems of Irrational Invariants of Associated Point Sets.

By Clyde M. Huber.

Introduction. The theory of invariants of algebraic forms has been developed to a great extent by the symbolic methods of Aronhold and Clebsch along the line of rational integral invariants, rational in the domain of the coefficients of the given form. In the early days expressions in terms of the differences of the roots were used and the English writers (as in Elliott's treatise) developed the theory to a well organized form but more particularly from the standpoint of obtaining rational integral invariants of a binary form in terms of irrational invariants (rational in the domain of the roots of the form). The products of the differences of the roots of a binary sextic were used by Joubert * and Richmond † to obtain rational integral invariants of the sextic but not to prove the completeness of the system. The completeness of the system was established from this point of view by A. B. Coble ‡ in a paper to which reference will be made frequently. Miss Whelan, in *American Journal of Mathematics,* Vol. 48 (1926), p. 73, obtained rational integral invariants (rational in the domain of the coefficients) of the binary octavic in terms of the irrational invariants.

In several papers, references to which are given in an address § at a meeting of the Chicago Section of the American Mathematical Society at Cincinnati, December 28, 1923, Professor Coble points out the connection between linear systems of irrational invariants of two types of point sets in S_1 and S_p respectively and the solution of the equation of degree $2p+2$. Numerous geometric and algebraic applications of the irrational invariants are made in these papers as well as the outline of their connection with hyperelliptic modular functions of genus p.

* Joubert, "Sur l'équation du sixième degré," *Comptes Rendus,* tome 64 (1867).

† H. W. Richmond, "Note on the Invariants of a Binary Sextic," *Quarterly Journal of Mathematics,* Vol. 31 (1899), p. 57.

‡ A. B. Coble, "Point Sets and Allied Cremona Groups," *Transactions of the American Mathematical Society,* Vol. 16 (1915), p. 155.

§ A. B. Coble, "The Equation of the Eighth Degree," *Bulletin of the American Mathematical Society,* Vol. 30 (1924), p. 301.

It is the purpose of this paper to investigate the linear systems of irrational invariants with the object of determining complete systems of such invariants. In § 1 a résumé of some of the elementary notions and definitions used throughout the paper are given. Section 2 gives a method of expressing irrational invariants of a class called *polycyclic* in terms of a class called *monocyclic* and in case of the even degree equation the polycyclic invariants are expressed in terms of products of the differences of the roots linear in each root, that is, in terms of *linear* invariants. In § 3 it is found that if all invariants of degree 4 are expressible in terms of invariants of degree 2 for an odd degree equation, the invariant of any degree for the odd degree equation may be expressed in terms of those of degree 2, i. e. a complete system is made up of a number of linearly independent *cyclic* invariants. The main object of the paper is then somewhat interrupted to give applications of this theorem to the quintic and septimic in §§ 4, 5 respectively as well as a discussion for the case of the septimic of a general mapping problem. In § 6 the principal aim of the paper is accomplished, namely to prove that for the set of points $P^1{}_{2p+2}$ defined by a binary equation of even degree, a complete system of irrational invariants is made up of linearly independent linear invariants. For n odd it would appear that a complete system is made up from the cyclic invariants; however, the writer has succeeded in establishing this for specific cases only. A comparatively simple method of exhaustion is given whereby the conjectured theorem can be verified in any specific case. In the last section invariants of a linear system on a set of points in S_p are expressed in terms of the linear invariants of a set of points in S_1, each set defined, to within projective modification, by the binary equation of degree $2p+2$.

1. *Preliminary Definitions and Remarks.* The general form of a rational integral invariant of a set of points $P^1{}_n$ on a line, which may be regarded as the roots of a binary form of order n, has been generalized by Professor Coble ‡ as follows: A rational integral invariant of the point set $P_n{}^\kappa$ of n points in S_κ is a rational integral function of the $n/(\kappa+1)$ determinants $P(i_1, i_2, \cdots, i_{\kappa+1})$ which is (A) homogeneous and of the same degree in the coordinates of the n points and (B) unaltered in value however the points be permuted. An irrational invariant of $P_n{}^\kappa$ is a function which satisfies all of the requirements of the foregoing definition except (B). An incomplete integral invariant (rational or irrational) of $P_n{}^\kappa$ is one which satisfies the proper requirements for a set $P_{n'}{}^\kappa$ $(n'<n)$ contained in $P_n{}^\kappa$.

For a set of n points in S_1 such a form is in general

$$(1) \qquad I = \sum \prod (12)^{a_{12}} (13)^{a_{13}} (14)^{a_{14}} \cdots (\overline{n-1\, n})^{a_{n-1,n}}$$

where (ij) is an abbreviation for the determinant $p_{i1}p_{j2} - p_{i2}p_{j1}$ and p_{i1}, p_{i2} $(i = 1, 2, \cdots, n)$ are the coordinates of the points of the set $P^1{}_n$, and the a_{ij}'s are positive integers, or zero such that

$$(2) \qquad \sum_{j=1}^{n} a_{ij} = N, \quad (i = 1, 2, \cdots, n), \quad a_{ij} = a_{ji}, \quad a_{ii} = 0.$$

N is the degree of the invariant in the coordinates of each point.

In the same paper it is shown that for every invariant of any type of $P_n{}^\kappa$ there is obtained an invariant J of the same type of the associated set $Q_n{}^{n-\kappa-2}$ and the two are equal to within a factor which is a power, whose index is the weight, w, of I equal to $nN/(\kappa + 1)$ of an undetermined constant. To a complete system of invariants I, there corresponds a complete system of invariants J.

To find a complete system of irrational invariants of the set of points $P^1{}_n$ and hence of its associated set $Q_n{}^{n-3}$, we have then to solve the system of diophantine equations (2). The different possible types of independent solutions of this system then yield the different types of products π entering into the invariant I.

Now it is known that in the case of $P^1{}_4$ and $P^1{}_6$ complete systems of irrational invariants are made up of certain elementary products. For $P^1{}_4$ a complete system is $(12)(34)$, $(13)(24)$, $(14)(23)$ only two of which are linearly independent. For the set $P^1{}_5$ we shall find later (§ 4) that the complete system is made up from the invariants of degree two, called cyclic invariants, which are of the following types:

$$(3) \qquad (ij)(jk)(kl)(lm)(mi);$$
$$(ij)^2(kl)(lm)(mk).$$

All products π entering into the invariant I of $P^1{}_5$ are expressible directly as products of factors of the type (3).

The first product of (3) we designate as *monocyclic,* that is consisting of one cycle. The general *polycyclic* invariant with k cycles for the set $P^1{}_n$ is an invariant of degree two in the coordinates of each point which has the following form:

$$(4) \qquad \{(a_1a_2)(a_2a_3)\cdots(a_{\alpha-1}a_\alpha)(a_\alpha a_1)\}\{(b_1b_2)(b_2b_3)\cdots(b_\beta b_1)\}\cdots$$
$$\{(k_1k_2)\cdots(k_\lambda k_1)\},$$

where $\alpha+\beta+\cdots+\lambda=n$. The number of types of cyclic invariants for P^1_n is then the number of different partitions of n, including of course the number n, and exclusive of the number one, since from the definition we must have the coordinates of at least two points in any single cycle.

A linear invariant we define as a product linear in the coordinates of each point and hence of the form $(ij)(kl)\cdots(rs)$. Such linear invariants clearly exist only in the cases in which n is even. Moreover it is evident that for n odd we can have no invariant of odd degree, since the sum of all the a_{ij}'s in the system (2) is even and on the right the sum would give an odd multiple of N, hence N itself would have to be even. The treatment of the two cases will be different and the results are of a different character.

The complete systems for P^1_6 and P^2_6 have been determined.* For P^1_6 the product π can be expressed rationally and integrally by the use of the binary determinant identity $(12)(34)=(13)(24)+(14)(32)$ in terms of the fifteen products of the type $(ij)(kl)(mn)$ of which only 5 are linearly independent.

2. *Expression of Polycyclic Invariants in Terms of Monocyclic Invariants in General.* We now show how to express the general polycyclic invariant on any number of roots in terms of monocyclic invariants. Such a polycyclic invariant may be written in the form (4). It is sufficient to show that two adjacent cycles may be combined into a sum of monocycles, since this process could be repeated as long as two or more cycles are left in any single term of the expression for the original polycyclic invariant. Consider any two adjacent cycles without any restriction as to the number of roots occurring in either cycle (save that the number in both shall not exceed n) as the following:

$$\{(a_1a_2)(a_2a_3)\cdots(a_{\alpha-1}a_\alpha)(a_\alpha a_1)\}\ \{(b_1b_2)(b_2b_3)\cdots(b_{\beta-1}b_\beta)(b_\beta b_1)\}.$$

Now apply the binary determinant identity to the product $(a_\alpha a_1)(b_1b_2)$. We then form a sum of two cycles as follows:

$$(a_1a_2)(a_2a_3)(a_3a_4)\cdots(a_{\alpha-1}a_\alpha)(a_\alpha b_1)(b_1b_\beta)\cdots(b_3b_2)(b_2a_1)+(a_1a_2)(a_2a_3)\cdots(a_{\alpha-1}a_\alpha)(a_\alpha b_2)(b_2b_3)\cdots(b_{\beta-1}b_\beta)(b_\beta b_1)(b_1a_1).$$

Proceeding in this way as long as two adjacent cycles remain we eventually obtain only monocyclic products.

We now show that in case n is even, that the general cyclic invariant can be expressed in terms of linear invariants. We first express the cyclic invariant in the above manner in terms of monocyclic invariants containing

* See "Point Sets" paper of Coble, referred to above.

an even number of roots. It is evident that every monocyclic invariant containing an even number of roots can be expressed as a product of two linear factors as follows:

$$(a_1a_2)(a_2a_3)(a_3a_4)(a_4a_5)\cdots(a_{2p-1}a_{2p})(a_{2p}a_{2p+1})(a_{2p+1}a_{2p+2})(a_{2p+2}a_1)=$$
$$\{(a_1a_2)(a_3a_4)\cdots(a_{2p-1}a_{2p})(a_{2p+1}a_{2p+2})\}$$
$$\{(a_2a_3)(a_4a_5)\cdots(a_{2p}a_{2p+1})(a_{2p+2}a_1)\}.$$

Hence the general cyclic invariant on an even number of points is a rational, integral function of the linear invariants.

THEOREM I. *The general polycyclic invariant* (4) *for the point set* P^1_n *can be expressed as a sum of monocyclic invariants by use of the binary identity. If* $n=2p+2$ *the general cyclic invariant can be expressed rationally and integrally in terms of the linear invariants.*

3. *Conversion of any Invariant Product* π *into a Reducible Product* π'. Consider now any product π of degree N, for any value of n. If any a_{ij}, say without loss of generality a_{12}, equals N the indices 1 and 2 will not occur again in π and the remaining product will be an invariant for a set of points contained in P^1_n. Then if the invariants of the set P^1_{n-2} are reducible, the invariant product π for P^1_n is reducible. We can easily verify that a complete system for the set P^1_3 is the single variant (12)(23)(31) by a direct solution of the diophantine system (2) for the value $n=3$. Moreover we know that all invariants of the set P^1_4 are reducible as referred to previously. If then we can take any product π and carry out on its exponents a series of operations which modify them so that eventually one exponent becomes equal to N, then, by induction, the modified product π' is reducible.

Suppose then that every a_{ij} is less than N, and let a_{ij} and a'_{ij} be any two solutions of the diophantine system

$$\begin{array}{lrl} & a_{12}+a_{13}+a_{14}+\cdots+a_{1n} & =N, \\ & a_{21}\qquad+a_{23}+a_{24}+\cdots+a_{2n} & =N, \\ (6) & a_{31}+a_{32}\qquad+a_{34}+\cdots+a_{3n} & =N, \\ & \cdots\cdots\cdots\cdots\cdots\cdots & \\ & a_{n1}+a_{n2}+a_{n3}+a_{n4}+\cdots+a_{n,n-1} & =N, \end{array}$$

for the same value of N.

Set $\delta = a_{ij} - a'_{ij}$, so that the δ's satisfy the system

$$
(7) \qquad \begin{array}{l}
\delta_{12} + \delta_{13} + \delta_{14} + \cdots + \delta_{1n} = 0, \\
\delta_{21} \qquad + \delta_{23} + \delta_{24} + \cdots + \delta_{2n} = 0, \\
\delta_{31} + \delta_{32} \qquad + \delta_{34} + \cdots + \delta_{3n} = 0, \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot, \\
\delta_{n1} + \delta_{n2} + \delta_{n3} + \delta_{n4} + \cdots + \delta_{n,n-1} = 0.
\end{array}
$$

Adding the last $n-1$ equations and subtracting from this result the first equation, we have

$$
(8) \qquad \delta_{23} + \delta_{24} + \delta_{25} + \cdots + \delta_{2n} + \delta_{34} + \cdots \\
+ \delta_{3n} + \delta_{45} + \cdots + \delta_{n,n-1} = 0,
$$

which we may write as

$$
(8') \qquad \delta_1 + \delta_2 + \cdots + \delta_{(n-1)(n-2)/2} = 0.
$$

It is evident that all solutions of this equation are compounded from these elementary solutions

$$
\delta_i = 1, \; \delta_j = -1, \quad (i, j = 1, 2, \cdots, (n-1)(n-2)/2), \quad (i \neq j), \\
\text{all other } \delta\text{'s} = 0.
$$

But these are not all independent since from $n(n-3)/2$ like $\delta_1 = 1$, $\delta_j = -1$, $[j = 2, 3, \cdots, (n-1)(n-2)/2]$ the remaining elementary solutions can be obtained. These solutions lead in (8) to two distinct types of solutions, namely the types $\delta_{23} = 1$, $\delta_{24} = -1$ in which the δ's have a common subscript and the type $\delta_{23} = 1$, $\delta_{45} = -1$ in which the δ's have no common subscript. Hence for the solutions of the system (7) we obtain the two types

$$
(9) \qquad \begin{array}{ll}
\text{Type I:} & \left.\begin{array}{l} \delta_{23} = \delta_{41} = 1 \\ \delta_{24} = \delta_{31} = -1 \end{array}\right\} \text{ all other } \delta\text{'s} = 0, \\
\text{Type II:} & \left.\begin{array}{l} \delta_{23} = \delta_{14} = \delta_{15} = 1 \\ \delta_{45} = \delta_{12} = \delta_{13} = -1 \end{array}\right\} \text{ all other } \delta\text{'s} = 0.
\end{array}
$$

The difference between the exponents of like factors of any two products π for the same value of N can then be expressed in terms of elementary differences of two types. Each of these types of elementary differences then leads to an operation on the exponents which we denote by O_1 and O_2. O_1 is an operation which reduces a_{ij} and a_{kl} by one and increases a_{ik} and a_{jl} by one. O_2 is an operation which reduces a_{ij}, k_{kl}, a_{km} by one and increases a_{kl}, a_{kj}, a_{lm} by one.

We observe that the operation O_2 can be expressed as a product of operations of type O_1, which may be shown as follows. The operation of type O_1 replaces the product $(ij)(kl)$ by the product $(ik)(jl)$; the operation of type O_2 replaces the product $(ij)(kl)(km)$ by the product $(ik)(kj)(lm)$. Let a specific operation of the type O_1 be $O_{ik,jl}$ which replaces $(ik)(jl)$ by $(ij)(kl)$. If we operate upon $(ik)(jl)(km)$ with $O_{ij,kl}$ we obtain the product $(ik)(jl)(km)$. Now we operate with $O_{jl,km}$ upon the resulting product and obtain $(ik)(jk)(lm)$, which gives the same result as if we had operated on the original product with the operation $O_{ij,kl,km}$ of type O_2, or

$$O_{ij,kl,km} = O_{ij,kl} \cdot O_{jl,km}\,.$$

THEOREM II. *Operations of type O_2 can be replaced by products of operations of type O_1.*

Since however only positive or zero values of the a_{ij}'s are in question, these operations are to be applied only to such products as will yield products with positive or zero exponents.

For a given value of N let π be the product

$$(12)^{\alpha}(13)^{\beta}(14)^{\gamma}\cdots(\overline{n-1,}\,n)^{\sigma},$$

where we now assume $\alpha, \beta, \cdots, \sigma < N$. Now the indices 1 and 2 must occur again in the product and suppose that each occurs with a different index, say with 3 and 4 respectively. We can by an operation of type O_1 increase the exponent of (12) and at the same time decrease the exponents of the factors (13) and (24). We may then by repeated applications of operations of type O_1 increase the degree of the factor (12) and decrease the degrees of the other factors containing the indices 1 and 2 so long as 1 and 2 each occur again in factors with different roots. We can by a finite number of applications of operations of type O_1 come to the stage where either (a) the degree of the factor (12) is N, or (b) we have 1 and 2 each occurring with the same index, say 3.

In case (a) the product π' would then be reducible.

If case (b) occurs we will have a product of the form

$$(12)^{r}(13)^{s}(23)^{t}\cdots(\overline{n-1}\,n)^{u},$$

where $r+s=N$, and $r+t=N$, from which it follows that $t=s$, but we may not have $s+t=N$. If the latter were true we would have the cyclic factor $(12)^{v}(23)^{v}(31)^{v}$ and the remaining factor would be an invariant on $n-3$ of the points and we would then be led to a reducible product π'. This

clearly can occur only in case N is even. Suppose then that $s+t<N$. By means of the operation $O_{13,23,lm}$ we can decrease the exponents of the factors (13) and (23) and at the same time the exponent of some other factor, say (45), containing none of the indices 1, 2, or 3. Now such a factor must clearly be of degree greater than $s+t$, or there must be at least $s+t$ such factors that do not contain the indices 1, 2, or 3. Hence at no time will there be negative exponents introduced by s applications of the operation $O_{13,23,45}$. At the same time we will be increasing the degree of the factor (12), until eventually its degree, a_{12}, will be equal to N, the degree of the invariant π. Hence by previous argument, the resulting product π' is expressible in terms of the cyclic or linear invariants, or π' is reducible.

THEOREM III. *Every invariant product π for the point set P^1_n can be converted by means of the operations of type O_1 into a sequence of products $\pi, \pi^{(1)}, \pi^{(2)}, \cdots, \pi^{(r-1)}, \pi^r=\pi'$ in which π' is a reducible product.*

If further we can show that the *reverse* of the above series of operations applied to a reducible product always leads to a reducible product, then the original product upon which we operated must have been reducible. We have shown that any operation of type O_2 can be expressed as a product of operations of type O_1. Moreover the reverse of an operation of type O_1 can be applied to a single cyclic invariant, or to a product of two cyclic invariants; also to a single linear invariant, to a product of two linear invariants, or to a product of a linear and a cyclic invariant. That is, the operations of type O_1 can affect at most in a single application an invariant product of degree four. If then we show that all solutions of the system (6) for N equal to 3, and 4 are reducible, then the product π is reducible for any value of N.

This proceedure can be further shortened by a separation of the cases in which n is odd or even. We first consider the case in which n is odd giving two methods of proving the reducibility of any product π for the set P^1_n for this case. We defer the treatment of the even case until § 6. It is evident that the reverse of an operation O_1 applied to a single cyclic product yields such a product; moreover, there are no invariants of degree 3 for n odd. (1) Hence it is sufficient to show that the reverse of operations of type O_1 applied to a product of two cyclic invariants always yields a product of cyclic invariants. For in that case in recovering the original product π we would pass back through a sequence of reducible products at each operation and hence the original product π would be reducible. (2) It is sufficient to show that all invariants of degree 4 are reducible, as in that case the application of the reverse of operations of type O_1 would always yield invariants of degree

4 which by hypothesis are reducible and therefore by the first alternative the original product π must be reducible. Both of these conditions are obviously necessary.

THEOREM IV. *A necessary and sufficient condition for the reducibility of any invariant product π for the point set $P^1{}_n$ when n is odd is that all invariants of degree 4 shall be reducible.*

It will be observed that the above two methods are each equivalent to Theorem IV, however the proceedure for verification in the two cases is quite different. In the following two sections we shall apply the first of the above methods to determine the complete system for the set $P^1{}_5$ and the second method to determine the complete system for the set $P^1{}_7$. We shall show in each case that the cyclic invariants form a complete system of irrational invariants.

4. *Complete System for $P^1{}_5$.* For the set $P^1{}_5$ we have two types of cyclic invariants, namely, $(ik)^2(jl)(lm)(mj)$ and $(ik)(kj)(jl)(lm)(mi)$. The reverse of an operation of type O_1 has the effect of interchanging two roots one of which we assume occurs in the first invariant of a product of two cyclic invariants, and the other in the second invariant factor. Such an interchange sends the product $(ik)(jl)$ into the product $(ij)(kl)$. With respect to such an operation the product of two cyclic invariants may be affected as follows:

$$
(10)\qquad
\begin{aligned}
&(1).\quad (ik)^2(i_3i_4i_5)\cdot(jl)^2(j_1j_2j_5),\\
&(2).\quad (ik)^2(i_3i_4i_5)\cdot(j_1j_2)^2(jlj_5),\\
&(3).\quad (ik)^2(i_3i_4i_5)\cdot(jlj_1j_2j_5),\\
&(4).\quad (i_3i_4)^2(iki_5)\cdot(j_1j_2)^2(jlj_5),\\
&(5).\quad (i_3i_4)^2(iki_5)\cdot(jlj_1j_2j_5),\\
&(6).\quad (iki_3i_4i_5)\cdot(jlj_1j_2j_5),
\end{aligned}
$$

$$\text{where } (i_li_mi_n) = (i_li_m)(i_mi_n)(i_ni_l).$$

The effect of the interchange of the indices j and k may be observed in each of these products, noting that the resulting product is reducible in each case. We illustrate the type of verification for the first case.

We may without loss of generality take $(ik)(jl)$ as the product $(12)(34)$.

(1). $(12)^2(i_3i_4i_5)(34)^2(j_1j_2j_5)$ goes by the above interchange into the product $(13)(12)(i_3i_4)(i_4i_5)(i_5i_3)(24)(34)(j_1j_2)(j_2j_5)$ (j_5j_1). Now the index 3 must occur in the second cycle of the first factor and similarly the index 1 must occur in the second cycle of the second factor. We may take each

of these indices in the leading position in the cycle, so that the modified product has the form $(13)(12)(3i_4)(i_4i_5)(i_53)\cdot(24)(34)(1j_2)(j_2j_5)(j_51)$. The factor (i_4i_5) must be either (45) or (54) and (j_2j_5) must be either (25) or (52). In either case we have the cyclic factor $(12)(25)(54)(43)(31)$, so that the remaining factor must necessarily be cyclic, and hence the product is reducible after the application of the reverse of an operation of type O_1.

The argument may be carried out in a similar manner for the remaining cases as the reader may verify and need not be given here. The reverse of the operations required to modify π so that it becomes a product π' which we know to be reducible then yields reducible products at each stage, so that the original product must have been reducible. A complete system of irrational invariants for P^1_5 is made up from products of the types

$$(11)\qquad \begin{aligned}&(a).\quad (ij)(jk)(kl)(lm)(mi);\\ &(b).\quad (ij)^2(kl)(lm)(mk).\end{aligned}$$

The invariants of type (b) are expressible by Theorem I as sums of invariants of type (a), and conversely. It is shown in the "Point-Sets" paper of Coble (p. 190) referred to above that there are just six products of either type that are linearly independent.

THEOREM V. *A complete system of irrational invariants for the set* P^1_5 *consists of six linearly independent products of the type* (11, a).

These six linearly independent products map binary quintics and thus points of the plane upon a quintic two way in S_5. This is a fairly well-known surface being also the map of the plane by a linear system of cubic curves with simple points at four base points in the plane.

5. *Complete System for* P^1_7. In this case we show that every invariant of degree 4 is reducible and hence by Theorem IV the general invariant must be reducible.

Consider the product π for the set P^1_7 of degree 4. If any $a_{ij}=4$, then the product π is reducible, since the remaining factor would be an invariant on the set P^1_5 contained in the set P^1_7 which we have shown to be reducible in § 4. Let us assume then that no exponent in the product π is equal to 4. We now list the possible types of products π in which we have three and no more exponents equal to 3, two exponents equal to 3, etc., showing in every case that π contains a cyclic factor, and hence every invariant of degree 4 for P^1_7 is reducible as required. It may be easily verified that these are the

only types, and that all other products obtained are either excluded (because of coming under a previous classification), or impossible (i. e. a case in which the factor (ii) would have to be introduced in order to complete the degree of the invariant in the coordinates of the point whose index is i). This method of verification we illustrate, the process being similar for other cases.

Let three exponents of π be equal to 3, we must then have a factor of the *type* $(12)^3(34)^3(56)^3$. We then consider the different possible ways of completing the product π in the coordinates of the point whose index is 1 to the required degree, then the point whose index is 2, etc., as follows:

$)^3(34)^3(56)^3$	(13)	(24)	(57)	(67)		requiring the factor (77) to complete;
		(27)	(45)	(67)		requiring the factor (77) to complete;
			(47)	(57)	(67)	has the cyclic factor $(12)^2(34)^2(56)(57)(67)$;
	(17)	(23)	(45)	(67)		an impossible type;
			(47)	(57)	(67)	has the cyclic factor $(12)^2(34)^2(56)(67)(75)$;
		(27)	(35)	(46)		impossible type;
				(47)	(67)	has the cyclic factor $(12)^2(34)(47)(76)(65)(53)$;
			(37)	(45)	(67)	has the cyclic factor $(12)^2(34)(45)(56)(67)(73)$.

In a similar way we could find the remaining types of products all of which contain a cyclic factor and hence are reducible. Now every product of degree four which is an invariant of the set $P^1{}_7$ must come under some one of the above types. But each product of any one of these types is reducible, therefore all products π of degree four are reducible. Hence, by Theorem IV it follows that any invariant product π for the set $P^1{}_7$ is reducible.

THEOREM VI. *Every invariant product π for the point set $P^1{}_7$ is reducible, that is, can be expressed as a product of the following types of cyclic invariants:*

$$
(12)\quad
\begin{array}{ll}
(12') & 360 \text{ of type } (ij)(jk)(kl)(lm)(mn)(no)(oi);\\
(12'') & 252 \text{ of type } (ij)^2(kl)(lm)(mn)(no)(ok);\\
(12''') & 105 \text{ of type } (ij)^2(kl)^2(mn)(no)(om);\\
(12^{IV}) & 105 \text{ of type } (ij)(jk)(ki)(lm)(mn)(no)(ol).
\end{array}
$$

By Theorem I each of the cyclic invariants of the last three types may be expressed rationally and integrally in terms of invariants of the first type, that is, the monocyclic type. A reduced complete system of irrational invariants for $P^1{}_7$ is then made up from certain linearly independent products out of the 360 products (12′). We now determine the number of linearly independent invariants in the complete system.

If in the 105 invariants of the third type in (12) we set $6 = x$, $7 = y$ we obtain the following types of irrational covariants of a binary quintic:

$$
(13) \qquad
\begin{array}{ll}
\text{(a)} & 10 \text{ of type } (xy)^2(12)^2(34)(35)(45); \\
\text{(b)} & 15 \text{ of type } (xy)(1x)(1y)(25)^2(34)^2; \\
\text{(c)} & 20 \text{ of type } (1x)^2(2y)^2(34)(35)(45); \\
\text{(d)} & 60 \text{ of type } (1y)^2(4x)(5x)(23)^2(45).
\end{array}
$$

These forms can be expressed in terms of powers of the determinant (xy) whose coefficients are elementary covariants and invariants arising from polars of the given forms in a unique manner.* In the expansion of the above forms there arise aside from numerical coefficients

(a′) 10 invariants of the type $(12)^2(34)(35)(45)$;

(b′) quadratic covariants of the types $(1x)^2(25)^2(34)^2$,
$(1x)(2x)(12)(34)(35)(45)$,
$(1x)(2x)(13)(24)(35)(45)$; $(1x)(2x)(13)(23)(45)^2$,

(c′) quartic covariants of the types $(1x)^2(2x)(3x)(23)(45)^2$,
$(1x)^2(2x)^2(34)(35)(45)$, $(1x)(2x)(3x)(4x)(15)(25)(34)$.

There are six linearly independent invariants of the type (a′) as noted above.

By use of the six functions of Joubert ‡ subject to the relation $A + B + C + D + E + F = 0$, the quadratic covariants can all be expressed in terms of the 15 products of the functions A, etc. These products are all linearly independent so that there are in all only 15 linearly independent quadratic covariants.

The quartic covariants can all be expressed in terms of the 30 quartic covariants of type $(1x)(2x)(3x)(4x)(15)(25)(34)$, which we designate as $q_{34,5}$. These are grouped in five sets of 6 each according to the root that is isolated, as in the set $q_{12,5}$; $q_{13,5}$; $q_{14,5}$; $q_{23,5}$; $q_{24,5}$; $q_{34,5}$. The q's in each set are subject to three relations of the type $q_{34,5} + q_{41,5} + q_{13,5} \equiv 0$, analogous to the relations on the six differences of the four indices 1, 2, 3, 4 from which we conclude that in each set of six covariants there are three linearly independent ones.

---

* A. Clebsch, *Theorie der binären Formen*, Leipzig, Teubner (1872), § 7.

There can be no other relations existing containing covariants from different sets of six for suppose there were some relation such as

$$(14) \qquad \lambda_1(2x)(3x)(4x)(5x) + \lambda_2(1x)(3x)(4x)(5x) + \cdots + \lambda_5(1x))2x)(3x)(4x) \equiv 0,$$

where the λ's represent any coefficients whatever. Now let $x = 1$ and all terms but the first in relation (14) vanish. Since in general the five roots are distinct, $\lambda_1 = 0$. Similarly we may show that the remaining λ's must be zero, that is, no relation of the type (14) could exist. The only linear relations satisfied by the 30 quartic covariants are those arising from the relations within the sets of six with one root isolated, hence there are just 15 linearly independent quartic covariants.

Furthermore there can be no relations existing among covariants of different order. Hence there are 36 linearly independent invariants and covariants. Since the 105 covariants are expressible in terms of these elementary ones we conclude that there are 36 linearly independent covariants of types (13).

Now replacing x and y by the roots 6 and 7 we have 36 linearly independent invariants of the type (12‴). These are all expressible in terms of those of type (12′) and conversely, hence the number of linearly independent invariants in the two systems is the same.

Theorem VII. *The simplest reduced complete system of irrational invariants of the point set $P^1{}_7$ consists of* 36 *linearly independent invariants from the set of* 360 *invariants* (12′).

It would seem that in general the invariants of the odd degree equation are expressible in terms of the monocyclic invariants. The methods sketched above will suffice to verify this in any particular case. We now give for the particular case of $P^1{}_7$ a map determined by the irrational invariants.

The 36 linearly independent cyclic invariants for $P^1{}_7$ map binary septimics upon a four way in S_{35}. We now determine the order of this spread which is invariant under the linear transformation group of order $7!$ in S_{35} corresponding to the symmetric group of the permutations of the roots of the binary septimic.

Let t_7, a root of the binary septimic, be transformed to infinity and $t_1 + t_2 + \cdots + t_6 = 0$. Every binary septimic determines a point in S_4, after the above transformation, which is then mapped upon the spread in S_{35} by the 36 linearly independent invariants. In S_4 we have certain spreads

defined by the cyclic invariants themselves. These spreads evidently have multiple points at the points of a set P_6^4 as follows:

$$\begin{array}{cccccc} -5 & 1 & 1 & 1 & 1 & 1, \\ 1 & -5 & 1 & 1 & 1 & 1, \\ 1 & 1 & -5 & 1 & 1 & 1, \\ 1 & 1 & 1 & -5 & 1 & 1, \\ 1 & 1 & 1 & 1 & -5 & 1, \\ 1 & 1 & 1 & 1 & 1 & -5. \end{array}$$

The multiplicity is of order 3 at each of these points, the order of the spread being 5. We then have quintic three ways in S_4, four of which intersect in a finite number of variable points, the number of which gives the order of the invariant spread in S_{35}. A line on two of the points of P_6^4 would cut the M_3^5 in six points hence the M_3^5 would contain the entire line, i. e., M_3^5 contains the 15 lines joining the points of P_6^4 as a fixed part.

Three of these M_3^5's would intersect in a curve of order 5^3 with multiple points of order 3 at the points of P_6^4, but the 15 lines are fixed, hence the order of the variable part will be $5^3 - 15$ with multiple points of order 22 at each point of P_6^4. A fourth M_3^5 would cut this variable curve in $5(5^3 - 15)$ points of which $6 \cdot 22 \cdot 3$ are fixed at P_6^4, hence there remain 154 variable points of intersection which gives the order of the invariant spread in S_{35}.

THEOREM VIII. *The 36 linearly independent invariants of the complete system for P^1_7 map projectively distinct binary septimics, and hence points of the linear space S_4 upon a spread of order* 154 *and dimension* 4 *in S_{35} which is invariant under the linear transformation group $G_{7!}$ which corresponds to the permutation group of the roots of the binary septimic.*

6. *Complete System for P^1_{2p+2}.* By Theorem III the product π, an invariant product for the point set P^1_{2p+2}, can be converted by a sequence of r operations described in § 3 into a product π' which is reducible. We have then a sequence of $r+1$ products $\pi, \pi^{(1)}, \cdots, \pi^{(r)} = \pi'$ such that π' factors into a product of cyclic and linear invariants. Moreover Theorem I enables us to express each of the cyclic invariant factors of π', by use of the binary determinant identity, as a sum of products of linear invariants. The product π' is then equal to a rational integral function Σ', of linear invariants like $(ij)(kl) \cdots (rs)$. We are thus led from the original invariant product π to a rational integral function Σ' of the linear invariants.

Now to recover from the product $\pi' = \Sigma'$ the original product π, we should apply the reverse of the operations necessary to convert π into the re-

ducible product π'. These operations by Theorem II are all expressible as products of operations of type O_1. If we show that the reverse of an operation of type O_1 applied to Σ' always yields a rational integral function of linear invariants, we would then have a sequence of the functions Σ such that $\pi' = \Sigma'$, $\pi^{(r-1)} = \Sigma^{(r-1)}$, $\cdots$, $\pi = \Sigma$. Hence the original product π would then be expressible as a rational integral function of linear invariants.

The operations of type O_1 have the effect of merely interchanging two indices j and k in a pair of factors of given type. In the application of the binary determinant identity to the cyclic invariant products in π' we can follow any pair of indices j and k and determine their position in the function Σ'. The interchange of the indices j and k in a given pair of factors of π' would then be equivalent in Σ' to the interchange of two definite indices j and k of each and every term of Σ'. In any term of Σ' such an interchange may occur within a single linear invariant, in which case a linear invariant would clearly result so that the reverse of O_1 would turn a rational function of linear invariants into such a function in this case. Or the indices may occur in different linear factors of the same term, in which case the resulting product will appear either as two linear factors or as a cyclic invariant which may be further expressed as a function of linear invariants by the method of § 2. Hence in any case the result of an operation interchanging a definite pair j, k in π', when carried out on the function Σ' yields a new $\Sigma^{(r-1)}$ which is again a rational integral function of linear invariants. We then have the equalities

$$\pi' = \Sigma',$$
$$\pi^{(r-1)} = \Sigma^{(r-1)}.$$

Now from $\pi^{(r-1)}$ we go to $\pi^{(r-2)}$ by an interchange of indices as described above, hence from $\Sigma^{(r-1)}$ we get a new $\Sigma^{(r-2)}$, etc., finally we get back to the original invariant product $\pi = \Sigma$. Hence every invariant product π for the even degree equation is equal to a rational integral function of the linear irrational invariants.

THEOREM IX. *A complete system of irrational invariants of the point set* P^1_{2p+2} *consists of a set of linearly independent products like* $(ij)(kl)\cdots(rs)$ *out of the* $\dfrac{(2p+2)!}{(p+1)!\,2^{p+1}}$ *such products.*

The number of such linearly independent invariants, which we call invariants of type (A), is $\dfrac{1}{p+2}\dbinom{2p+2}{p+1}$.*

* A. B. Coble, "The Equation of the Eighth Degree," *Bulletin of the American Mathematical Society*, Vol. 30 (1924), p. 302.

7. *Expression of Invariants* (B) *in terms of Invariants of Type* (A). The binary $(2p+2)$-ic defines three sets of points P^1_{2p+2} on S_1, Q^p_{2p+2} on a rational norm curve N^p in S_p, and R^{2p-1}_{2p+2} on N^{2p-1} in S_{2p-1}. The two extreme sets are associated sets,* and hence the invariants of R^{2p-1}_{2p+2} are proportional to the invariants formed from the complimentary determinants of the set P^1_{2p+2}. The set Q^p_{2p+2} gives rise to invariant products of the type

$$(1, 2, \cdots, p+1)(p+2, \cdots, 2p+2), \quad \text{where}$$

$$(15) \qquad (i_1 i_2 i_3 \cdots i_{p+1}) \equiv \begin{vmatrix} t_{i_1}^p & t_{i_1}^{p-1} & \cdots 1 \\ t_{i_2}^p & t_{i_2}^{p-1} & \cdots 1 \\ \cdot & \cdot & \cdot \\ t_{i_{p+1}}^p & t_{i_{p+1}}^{p-1} & \cdots 1 \end{vmatrix}$$

which we designate as an invariant of type (B). This determinant may also be expressed in the form

$$(16) \qquad (i_1 i_2 i_3 \cdots i_{p+1}) = \prod (i_r i_s); \quad r, s = 1, 2, \cdots, p+1; \quad r < s.$$

In this form we see that an invariant (B) is equal to an invariant of degree p of the set P^1_{2p+2}, which by Theorem IX of the preceding section is equal to a rational integral function of the linear invariants (A). We now give a method for expressing the invariants (B) in terms of the linear invariants (A).

The determinant product $(i_1 i_2 i_3 \cdots i_{p+1})$ can be arranged according to the following scheme:

$$(17) \qquad \begin{array}{r} (i_1i_2)(i_1i_3)(i_1i_4)(i_1i_5)(i_1i_6)\cdots(i_1i_{p-2})(i_1i_{p-1})(i_1i_p)(i_1i_{p+1}) \\ (i_2i_3)(i_2i_4)(i_2i_5)(i_2i_6)\cdots(i_2i_{p-2})(i_2i_{p-1})(i_2i_p)(i_2i_{p+1}) \\ (i_3i_4)(i_3i_5)(i_3i_6)\cdots(i_3i_{p-2})(i_3i_{p-1})(i_3i_p)(i_3i_{p+1}) \\ (i_4i_5)(i_4i_6)\cdots(i_4i_{p-2})(i_4i_{p-1})(i_4i_p)(i_4i_{p+1}) \\ (i_5i_6)\cdots(i_5i_{p-2})(i_5i_{p-1})(i_5i_p)(i_5i_{p+1}) \\ \cdots\cdots\cdots\cdots\cdots \\ (i_{p-3}i_{p-2})(i_{p-3}i_{p-1})(i_{p-3}i_p)(i_{p-3}i_{p+1}) \\ (i_{p-2}i_{p-1})(i_{p-2}i_p)(i_{p-2}i_{p+1}) \\ (i_{p-1}i_p)(i_{p-1}i_{p+1}) \\ (i_pi_{p+1}). \end{array}$$

Consider the product of all the factors on a diagonal beginning with the upper left and proceeding downward to the right. If we combine the first diagonal product with the last, we have a cyclic invariant, say C_1, since each

* See "Point Sets" paper of Coble, referred to above.

index occurs twice and only twice. Similarly, the second diagonal product with the next to the last one together form a cyclic invariant, etc. We now distinguish two cases, p odd and p even.

If $p = 2k + 1$, we have an odd number of diagonal products, the middle one of which in the above scheme is clearly a linear invariant on the $p + 1 = 2k + 2$ points. The remaining diagonal products combine in pairs to form cyclic invariants, so that we have

$$(i_1 i_2 i_3 \cdots i_{p+1}) = C_1 C_2 \cdots C_k \cdot L. \tag{18}$$

If $p = 2k$ we have an even number of diagonal products which combine as above in pairs into cyclic invariants giving

$$(i_1 i_2 i_3 \cdots i_{2k+2}) = C_1 C_2 \cdots C_k . \tag{18$'$}$$

Now the remaining determinant factor of the invariant (B) may be expressed as above in each case respectively as

$$(j_1 j_2 j_3 \cdots j_{2k+2}) = C_1' C_2' \cdots C_k' \cdot L',$$

or

$$(j_1 j_2 j_3 \cdots j_{2k+1}) = C_1' C_2' \cdots C_k'.$$

Any one of the C's multiplied by any one of the C''s gives a cyclic invariant on the set $P^1{}_{2p+2}$ which by the method of section 2 can be expressed rationally and integrally in terms of invariants (A). Similarly the remaining factors may be paired and expressed in terms of invariants (A). In case $p = 2k + 1$, the product $L \cdot L'$ is an invariant (A) so that the method holds for any invariant (B), giving in either case

$$(i_1 i_2 i_3 \cdots i_{p+1})(j_1 j_2 j_3 \cdots j_{p+1}) = R^p(A).$$

URBANA, ILLINOIS,
June 1, 1926.

Regular Maps and Their Groups.

By H. R. Brahana.

The geometrical representation of a finite group by means of fundamental regions, due to Dyck,* leads to a map that is transformed into itself by every operation of the group. The group may be represented transitively on symbols for regions, but in general it may not be represented transitively on symbols for edges or vertices. On the other hand it is well known † that the polyhedral groups may be represented transitively on symbols for regions, for edges, or for vertices of the regular polyhedra or the corresponding maps on a sphere. In the one case the number of regions of the map is equal to twice the order of the group and in the other the number of regions is equal to the order of the group divided by the number of sides of a region. Maps of the latter type we have called regular maps.

The polyhedral groups are the only ones that may be represented on regular maps on a sphere. There has been no attempt to determine the kinds of groups that may be represented on regular maps on surfaces of higher genus. Heffter ‡ showed that the metacyclic groups may be so represented. In a recent paper § all the maps of twelve five-sided regions with a group of order 120 containing an icosahedral subgroup were exhibited. In this last cited investigation the surfaces were allowed to be one-sided or two-sided and when the surface was two-sided a transformation of the map into itself in such a way as to reverse the sense of the boundary of each region was permitted. Lastly, the maps on a surface of genus one and the groups connected with them were considered.* It was shown that any such regular map must be made up of triangles, of quadrangles, or of hexagons and their groups must be generated by two operators of orders two and three with product of order

* "Gruppentheoretischen Studien," *Mathematische Annalen*, Vol. 20 (1882), pp. 1-44.

† Klein, *Vorlesungen ueber das Ikosaeder*, I, § 13.

‡ "Ueber metacyklische Gruppen und Nachbarconfigurationen," *Mathematische Annalen*, Vol. 50 (1897), pp. 261-268.

§ Brahana and Coble, "Maps of twelve countries, etc.", *American Journal of Mathematics*, Vol. 48 (1926), pp. 1-20.

* "Regular Maps on an Anchor Ring," *American Journal of Mathematics*, Vol. 48 (1926), pp. 225-240.

six, or by two operators of order two and four with product of order four, and conversely that to every such group there corresponds a regular map on an anchor ring.

In § 1 we recall a condition that has been stated previously as necessary that a given group be the group of a regular map and prove that it is sufficient by giving a method of constructing the map corresponding to the given group. In § 2 we note four classes of groups that satisfy the given condition and determine the genera of the surfaces on which the corresponding maps lie. In § 3 we apply the earlier considerations to determine the regular maps on a surface of genus two.

1. *The Group of a Regular Map.* A *map* is a finite set of distinct 0-cells, 1-cells, and 2-cells which constitutes a closed, connected, two-sided, two-dimensional manifold. We shall hereafter use the terms *vertices, edges* or *lines,* and *regions* for 0-cells, 1-cells, and 2-cells.

If the interior of a region of the map is imaged continuously on the interior of a circle there will be a finite * set of points on the boundary of the circle which correspond to the vertices of the map that lie on the boundary of the given region. The number of points in this set is the number of *vertices of the region.* The number of vertices of a region is the same as the number of *sides or edges of the region.*

A map of k n-sided regions will be said to be *regular* if the number of ways in which the surface can be put into $(1-1)$ correspondence with itself so that regions correspond to regions, edges to edges, and vertices to vertices, without reversal † of the sense of the boundaries of regions is kn.

The operations of transforming a map into itself in such a way as to preserve the sense of a region obviously constitute a group; this group will be called *the group of the map.* A map of k n-sided regions will be regular if its group is of order kn.

It is immediately evident and has been noted otherwhere * that the group of a regular map may be generated by two operators, viz.: S which leaves a given region fixed permuting its edges cyclically, and T which leaves an edge of this region fixed interchanging the two regions which have the edge in

* We exclude the case of a map defined by designating a single point on a sphere as the boundary of the remainder of the surface.

† It is well-known that the regions of a map on a two-sided surface may be sensed alike in two distinct ways, cf. Veblen and Young, *Projective Geometry,* Vol. II, p. 495. We require that there be kn ways of transforming k positively sensed regions into the same k positively sensed regions.

* Brahana and Coble, *loc. cit.,* p. 5.

common. T is of order two. By means of these two operations and combinations of them any region of the map may be transformed into any other region with a particular vertex of the first going into any vertex of the second.

In demonstrating the sufficiency of the above condition, we shall distinguish two cases: (1) T is not permutable with S or any group generated by a power of S; and (2) there exists a group generated by a power of S which is permutable with T. The tetrahedral, octahedral, and icosahedrol groups come within the first category and the dihedral groups are in the second.

We consider a group G generated by two operators S and T, the latter of order two and not permutable with any group generated by a power of the former. We distribute the operators of G in k right co-sets with respect to the subgroup H consisting of S and its powers and denote each co-set by a letter. Multiplication of all the operators of G on the right by any operator of G interchanges the co-sets and so determines a substitution on the k letters. The resulting substitution group will be transitive and will be simply isomorphic with G. We note for future use that the only substitutions which omit the letter corresponding to H are those which correspond to operators of H.

Let S_a, T_{ab}, and S_b be the substitutions corresponding to S, T, and TST respectively, b being the letter into which a is transformed by T_{ab}. The letter b appears in a cycle of S_a, otherwise S_b would omit a contrary to the hypothesis that T is not permutable with any subgroup generated by a power of S. The cycle C of S_a which contains b is of degree n, otherwise some power of S_a lower than the nth would leave b fixed; this would imply that $ST \cdot S^m = S^lT$ or $S^m = TS^{l-1}T$ which is impossible for the same reason.

We may now construct a regular map corresponding to G. A polygon of n sides may be denoted by a and bounded cyclically by n n-sided polygons named from C. The transform of a and C by T_{ab} gives b and its boundary. Continuing this process we get k bounded n-sided polygons. Since there are n substitutions leaving a fixed there will be n substitutions leaving b fixed and b will appear n times in the conjugates of C. Therefore, each region appears on the boundaries of n other regions. We may join the k polygons into a simply connected polygon in a plane with its edges paired in the ordinary manner. The two dimensional manifold so defined is two-sided, for any operation leaving a fixed is a power of S_a and so leaves C unchanged making it impossible to transform a into itself with its boundary reversed. Hence, *to every group in the first class there corresponds a regular map.*

The groups of the second category will be examined in two distinct classes

according as (a) T is permutable with S, or (b) T is not permutable with S but is permutable with a group generated by a power of S.

The groups of class (a) are cyclic and of even order if T is a power of S. A map corresponding to such a group is obtained by taking a polygon of n sides and joining opposite sides so as to give a two-sided surface.* An example of such a map is a four-sided region on an anchor ring; it contains one region, one vertex, and two edges.

If T is not a power of S the group is Abelian and of order $2n$. If there exists a regular map corresponding to it and containing an n-sided region it must contain two such regions. We may take two n-sided polygons and letter their sides $(abc \cdots f)$ and $(\alpha\beta\gamma \cdots \zeta)$ respectively. We join them together along a and α, making b correspond to β, c to γ, and so on, the members of a pair being oppositely sensed with respect to the double polygon. Such a map admits the operations $(abc \cdots f)(\alpha\beta\gamma \cdots \zeta)$ and $(a\alpha)(b\beta)(c\gamma) \cdots (f\zeta)$. It may be readily verified that the number of vertices is 1 or 2 according as n is odd or even and that the genus of the resulting surface is $(n-1)/2$ or $(n-2)/2$.

In the groups of class (b) the subgroup 1, T is not invariant and we may represent the group as a substitution group on symbols for the co-sets with respect to this subgroup by the method used for groups of the first category. If a is the letter corresponding to the set 1, T the substitution corresponding to S contains a in a cycle C of n letters, for the n co-sets $\begin{matrix} 1 & S & S^2 & \cdots S^{n-1} \\ T & TS & TS^2 & \cdots TS^{n-1} \end{matrix}$ are all distinct. If we denote the order of G by kn as before, we see that the substitution group is of degree $kn/2$. The number of letters in C and its conjugates is kn so that each of the letters appears twice. We may construct a map corresponding to G by taking an n-sided polygon for each of the conjugates of C and bounding it according to the letters of the conjugate. This time we name edges on the boundary instead of regions across the boundary. The polygons may be joined into a single two-sided surface by coalescing like-named edges with the usual precautions as to the senses of corresponding edges.

We note that the dihedral groups are contained in this class. If T transforms S into its inverse the group is dihedral. The resulting map contains two n-sided regions and lies on a sphere. It may be obtained by drawing an

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* The only requirement is that the pair of corresponding sides a and a' be oppositely sensed on the boundary of the polygon and that they be then joined so that the two senses coincide. See Brahana, "Systems of Circuits on Two-Dimensional Manifolds," *Annals of Math.*, Vol. 23 (1922), p. 146.

n-sided polygon on a sphere. T may transform S into some power of itself other than its inverse in which case the group is again of order $2n$ and the map consists of two n-sided polygons not on a sphere. If neither of the above conditions obtain the map still resembles the dihedral maps in that each region touches some other region along more than one edge. To see this let S_α and S_β be generators of the groups leaving the neighboring regions α and β fixed, and let $T_{\alpha\beta}\, S_\alpha\, T_{\alpha\beta} = S_\beta$, where S_α and $T_{\alpha\beta}$ are the substitutions corresponding to S and T respectively. Since $T\, S^m\, T = (S^m)^r$ for some m less than n, then $T_{\alpha\beta}\, S_\alpha{}^m\, T_{\alpha\beta} = (S_\alpha{}^m)^r = (S_\beta)^m$ leaves both α and β fixed and since $mr \neq n$ S_α must transform an edge common to α and β into another edge common to α and β.

We return from the digression of the last paragraph to state the principal result so far obtained in the following theorem:

A necessary and sufficient condition that G be the group of a regular map is that G be generated by two operators of which one is of order two.

2. *Some Types of Group that Give Regular Maps. The Symmetric and Alternating Groups.* It is well known that the tetrahedral, octahedral, and icosahedral groups are simply isomorphic with the alternating and symmetric groups of degree four and the alternating group of degree five. We extend the above result by means of two theorems, of which the first is:

To the symmetric group of degree n there corresponds a regular map of $(n-1)!$ *n-sided regions on a surface of genus*

$$p = 1 + (n-2)!\ (n^2 - 5n + 2)/4.$$

The existence of the map follows from the theorem of §1 and a theorem due to Moore * that the symmetric group of degree n is generated by two operators of orders n and two whose product is of order $(n-1)$. In order that we may determine the genus of the surface on which the map lies we shall recall a theorem that was used in *Regular Maps on an Anchor Ring.* In that paper it was proved (p. 227), though not explicitly stated, that

If S generates the group leaving a region fixed and T is the operator leaving an edge of the same region fixed, then ST generates the group leaving a vertex of the region fixed.

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* *Proceedings of the London Mathematical Soc.*, Vol. 28 (1896), pp. 357-366. See also Carmichael, *Quarterly Journal,* Vol. 49 (1922), p. 235.

From this theorem is follows that the number of regions at a vertex in a regular map is equal to the order of ST. In the maps under consideration there are thus $(n-1)$ regions at a vertex. The number of vertices is $n(n-2)!$, the number of edges is $n!/2$, and the number of regions is $(n-1)!$. From the Euler formula we obtain the genus $p=1+(n-2)!\,(n^2-5n+2)/4$. We give in the following table a list of the maps described by the theorem for $n=4$, 5, 6, and 7.

n	k	p	n	k	p
4	6	0	6	120	49
5	24	4	7	720	481

The first of these maps is the cube; the second is one of the maps described in *Maps of Twelve Five-Sided Regions,* etc. (*l. c.,* p. 19) and is the doubly covered figure II of that paper.

The alternating groups of degree greater than three are among the groups that give regular maps, for each group is generated by two operators one of which is of order two.* The generators may be chosen to be of orders 2 and $(n-1)$ with product of order $(n-1)$ if n is even, and of orders 2 and $(n-2)$ with product of order n if n is odd. When n is even we have $n(n-2)!/2$ vertices, $n!/4$ edges, and $n(n-2)!/2$ regions. The Euler formula takes the form

$$n(n-2)!/2-n!/4+n(n-2)!/2=2(1-p),$$

whence

$$p=1+n(n-2)!\,(n-5)/8.$$

When n is odd we have $(n-1)!/2$ vertices, $n!/4$ edges, and

$$n(n-1)(n-3)!/2$$

regions. Hence,

$$p=1+(n-1)(n-3)!\,(n^2-6n+4)/8.$$

We have the following theorem:

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* It may be shown readily that $S=(a_1, a_2, \cdots, a_{n-1})$ and $T=(a_1a_2)(a_3a_n)$ for n even, and $S=(a_1, a_2, \cdots, a_{n-2})$ and $T=(a_1a_{n-1})(a_2a_n)$ for n odd, generates the alternating group of degree n.

To the alternating group of degree $n(>3)$ *there corresponds a regular map of* $n(n-2)!/2$ $(n-1)$*-sided regions or a map of* $n(n-1)(n-3)!/2$ $(n-2)$*-sided regions on a surface of genus* $1+n(n-2)!(n-5)/8$ *or* $1+(n-1)(n-3!)(n^2-6n+4)/8$ *according as n is even or odd.*

It is of some interest to note the genera of the maps for small values of n. In the following list n' is the number of sides of each region of the map given by the theorem above.

n	n'	k	p	n	n'	k	p
4	3	4	0	6	5	72	19
5	3	20	0	7	5	504	199.

The first two are the tetrahedron and the icosahedron respectively.

Every regular map determines a second regular map which we shall call its *dual.* The dual of a map is obtained by taking a point within each region and joining the points of every pair of neighboring regions by an arc across their common edge, or by an arc across each common edge if more than one exists, the arcs being chosen so that no two intersect. The resulting map has a region for each vertex and a vertex for each region of the original map; the number of edges is the same in both. The cube and the octahedron are dual to each other, as are also the dodecahedron and the icosahedron. The dual of the tetrahedron is a tetrahedron; such a map will be called *self-dual.*

If S and T are the generators of the group from which a given map is obtained by the methods of § 1, the generators of the same group which would give the dual map are (ST) and T. The number of sides of a region of a map is equal to the number of regions at a vertex of its dual. Hence, *a necessary and sufficient condition that the map corresponding to the generators* S *and* T *be self-dual is that the orders of* S *and* ST *be the same.*

We have immediately the following theorem:

The maps given by the theorem concerning alternating groups are self-dual whenever n is even.

Subgroups of the Metacyclic Groups. The metacyclic group $G_{p(p-1)}$ of degree p (p must be prime) is generated by an element Σ of order p and a cyclic element S of order $(p-1)$.* The element Σ generates an invariant

* Netto, *Substitutiontheorie,* § 125.

subgroup and the remaining operators are the transforms of powers of S by Σ and its powers. S and Σ satisfy a relation $S^{-1}\Sigma S=\Sigma^m$, where $m^{p-1}\equiv 1$, mod p, and m satisfies no relation of the form $m^x\equiv 1$, mod p, where $x<p-1$. Since p is a prime (>2) S is of even order and $S^{(p-1)/2}$ is of order 2.

If $p-1$ contains any even factor ρ and $p-1=\rho\lambda$ then S^λ generates a cyclic group of order ρ which contains an element of order two. S^λ and Σ generate a group of order ρp. We shall show that this group $G_{\rho p}$ may be generated by $S'=S^\lambda$ and any of its elements of order two except $(S')^{\rho/2}$, e. g. by $T'=\Sigma^{-k}S'^{\rho/2}\Sigma^k$. From the fact that $S^{-1}\Sigma S=\Sigma^m$, we have $\Sigma S=S\Sigma^m$, and $\Sigma S'=S'\Sigma^{m^\lambda}$. Hence

$$T'=\Sigma^{-k}(S')^{\rho/2}\Sigma^k=(S')^{\rho/2}\Sigma^{-km^{(p-1)/2}}\Sigma^k=(S')^{\rho/2}\Sigma^{2k}.$$

This last relation is due to the fact that

$$m^{p-1}-1=(m^{(p-1)/2}-1)(m^{(p-1)/2}+1)\equiv 0, \text{ mod } p.$$

The invariant subgroup is generated by any power of Σ except identity and hence $[S',T']$ contains Σ. Therefore to every group $G_{\rho p}$ there corresponds a regular map.

In order to determine the genus of the surface on which the map corresponding to $G_{\rho p}$ lies we must find the order of $S'T'$. From considerations similar to those used above we see that

$$S'T'=(S')^{[(\rho/2)+1]}\quad\Sigma^{2k}$$

$$(S'T')^2=(S')^{2[(\rho/2)+1]}\quad\Sigma^{2k(m^{[(\rho/2)+1]\lambda}+1)}$$

$$=(S')^{2[(\rho/2)+1]}\quad\Sigma^{2k(-m^\lambda+1)}$$

$$(S'T')^3=(S')^{3[(\rho/2)+1]}\quad\Sigma^{2k(m^{2\lambda}-m^\lambda+1)}$$

and in general,

$$(S'T')^n=(S')^{n[(\rho/2)+1]}\quad\Sigma^{2k(1+m\lambda-m^{2\lambda}+\ldots+(-1)^{n-1}m^{(n-1)\lambda})}$$

If we set $(S'T')^n=1$ and seek the smallest value of n that will satisfy the relation, we shall have to find the smallest value of n that will satisfy the two congruences

(a) $\lambda n[(\rho/2)+1]\equiv 0$, mod $p-1$, and

(b) $2k(1-m^\lambda+m^{2\lambda}-m^{3\lambda}+\cdots+(-1)^{n-1}m^{(n-1)\lambda})\equiv 0$, mod p.

If n is odd (b) takes the form $\dfrac{1+m^{n\lambda}}{1+m^{\lambda}} \equiv 0$, mod p, the $2k$ being dropped because p is prime. This implies $m^{n\lambda} \equiv -1$, mod p, hence $n = \rho/2$ satisfies the congruence provided $\rho/2$ is odd. This value also satisfies (a) since $(\rho/2)+1$ is even and $n[(\rho/2)+1]$ is a multiple of ρ. If n is even (b) takes the form $\dfrac{1-m^{n\lambda}}{1+m^{\lambda}} \equiv 0$, mod p, in which case $m^{n\lambda} \equiv 1$ mod p, and $n = \rho$. This value obviously satisfies (a). Therefore the order of $S'T'$ is ρ or $\rho/2$ according as $\rho/2$ is even or odd.

The corresponding map will have p regions, $p\rho/2$ edges, and p or $2p$ vertices. From the Euler formula we find the genus of the surface. We state the result in the following theorem:

To every group of even order $p\rho$ which is contained in the metacyclic group of degree p there corresponds a regular map of p ρ-sided regions on a surface of genus $1+(p/4)(\rho-4)$ *or* $1+(p/4)(\rho-6)$ *according as $\rho/2$ is or is not a multiple of* 4.

Since when ρ is a multiple of 4 S' and $S'T'$ are of the same order, we have

The maps corresponding to $G_{\rho p}$ are self-dual whenever ρ is a multiple of 4.

When $\rho = 2$ the groups are dihedral and the maps lie on a sphere. When $\rho = p-1$ the groups are the metacyclic groups themselves and the maps are those given by Heffter (cf. the reference above). When $\rho = 4$ or 6 the maps lie on an anchor ring (cf. above).

It is worthy of note that of all the maps whose existence we have proved very few lie on surfaces of low genus. The Euler formula $V-E+F = 2(1-p)$ shows that if p is to be small V and F must be as large as possible, which for a group of given order requires that n and v, the orders of S and ST, be small. If we seek a map on a surface of low genus whose group is a subgroup of a metacyclic group the degree of the group or ρ or both must be small. If the genus is to be greater than 1 p must be at least 11 and ρ must be at least 8. The metacyclic group of degree 11 gives a map on a surface of genus 12; the map on the surface of lowest genus corresponding to a group G_{8p} is made up of 17 octagons on a surface of genus 18. The smallest map of decagons corresponds to a G_{31p} and lies on a surface of genus 32. We note in passing that whenever p is of the form $40h+1$ there exists a map of p octagons and a map of p decagons each on a surface of genus $p+1$. The

more interesting maps, i. e. those on surfaces of low genus, are all missing except some of the maps on an anchor ring.

Subgroups of the Modular Group. The two-rowed unit matrices with integer elements constitute a group simply isomorphic with the modular group.* If the elements of each matrix are reduced modulo n there is obtained a finite set of matrices of determinant 1, mod n, which constitute a group isomorphic $(1-\infty)$ with the modular group. This group, $G_{2\mu(n)}$, contains a single element of order two, viz.: $\begin{pmatrix} n-1 & 0 \\ 0 & n-1 \end{pmatrix}$, and so is not available for the group of a regular map. If, however, we make the further reduction of considering $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to be equivalent to $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ we obtain $G_{\mu(n)}$ which is in $(1-2)$ isomorphism with $G_{2\mu(n)}$. The group $G_{\mu(n)}$ is generated by the two operators $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ n-1 & 0 \end{pmatrix}$ which we denote by S and T. The order of S is n, the order of T is 2, and the order of ST is 3. The order of the group is given by the equation

$$\mu(n) = (n^3/2)(1-1/q_1^2)(1-1/q_2^2)\cdots \quad \text{where } n = q_1^{\gamma_1} q_2^{\gamma_2}\cdots$$

and the q's are distinct primes.

The group $G_{\mu(n)}$ determines a map of $\mu(n)/n$ n-sided regions, $\mu(n)/2$ edges, and $\mu(n)/3$ vertices. Using these values in the Euler formula we determine the genus of the surface. The result is the following theorem:

To every group $G_{\mu(n)}$ there corresponds a regular map of $\mu(n)/n$ n-sided regions on a surface of genus $1+(1/6-1/n)\mu(n)$.

It is evident that the maps associated by Klein-Fricke with the groups $G_{\mu(n)}$ have a close resemblance to the regular maps we have obtained. Their maps consist of $2\mu(n)$ triangles in which the subgroups of order n are represented by the $2n$ triangles that come together at a vertex of one type, the subgroups of order three by the six triangles at a vertex of another type, and the elements of order two by the four triangles at a vertex of a third type. We represent a subgroup of order n by a single n-sided region, which amounts to combining the $2n$ triangles at a vertex of the first type into a single region. If this combination is made at each of the vertices of the first type their

* The facts of this paragraph are to be found in Klein-Fricke, *Theorie der Elliptischen Modulfunktionen*, Leipzig (1890), Chapter 7.

maps will obviously become regular maps in the sense in which we are using the term.

For certain composite values of n $G_{\mu(n)}$ will contain distinct elements of the form $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$; for example, in addition to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ we have, when $n=8$, $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ and, when $n=15$, $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$. These elements will constitute an invariant subgroup of $G_{\mu(n)}$ and if this subgroup is of order m we obtain a group $G_{\mu(n)}/m$ in $(1-m)$ isomorphism with $G_{\mu(n)}$ by considering the operators of this invariant subgroup as equivalent. The groups $G_{\mu(n)}/m$ satisfy the condition of § 1 and so give regular maps.

The following is the list of maps corresponding to groups $G_{\mu(n)}$ for small values of n; it is essentially the combination of two lists given by Klein-Fricke.

k	n	p	k	n	p
4	3	0	24	7	3
6	4	0	24	8	5
12	5	0	36	9	10
12	6	1	36	10	13

The first four are regular polyhedra and a map on an anchor ring.

The subgroups of $G_{\mu(n)}$ for n prime are described by Klein-Fricke. Every such subgroup, and the metacyclic groups are among them, which is generated by two operators one of which is of order two gives a regular map. We shall not pursue this question further but shall note two groups of low order that give maps on surfaces of low genus. The group already mentioned obtained by taking $G_{\mu(8)}$ and considering $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ to be equivalent gives a G_{96} which corresponds to a map of 12 octagons on a surface of genus 3. If we take $G_{\mu(8)}$ and consider $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$. $\begin{pmatrix} 3 & 4 \\ 0 & 3 \end{pmatrix}$ and $\begin{pmatrix} 3 & 0 \\ 4 & 3 \end{pmatrix}$ equivalent we obtain a G_{48} which corresponds to a map of 6 octagons on a surface of genus 2. (Klein-Fricke, p. 652.) To see that the map is made up of octagons we note that $S=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is still of order 8, since $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ implies $a\equiv 0$, mod 8. We note for a later reference that $S^4T=TS^4$, where $T=\begin{pmatrix} 0 & 1 \\ 7 & 0 \end{pmatrix}$, for $TS^4T=\begin{pmatrix} 0 & 1 \\ 7 & 0 \end{pmatrix}\cdot\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} 0 & 1 \\ 7 & 0 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$.

3. *The Regular Maps on a Surface of Genus Two.* Our method of investigating this question is first to determine from the Euler formula the possible maps and then to examine the possibility of the existence of groups having the characteristics required by the maps. From the maps we are able to determine the order of the group, the orders of two generating operators, and the order of their product. Since the existence of a map implies the existence of a dual we may choose the generating operators of the group in two distinct ways whenever the proposed map is not self-dual. This is of appreciable advantage whenever the map is such that the generating operators of the group may be chosen so that S is of prime order, for it assures us that the order of the group must be twice the order of S if T is permutable with a group generated by a power of S, or else, in the opposite case, the group may be represented on symbols for regions of the proper one of the two dual maps. Having the order and degree of the group we are able to make use of the work that has been done in listing groups of low degree.

We shall consider first the cases of maps of one and two regions respectively. For a map of one region on a surface of genus two the Euler formula takes the following form $(n/v) - (n/2) + 1 = -2$ where v is the number of regions at a vertex and n is the number of sides of a region. This equation may be written in the form $(n-6)(v-2) = 12$. $(v-2)$ is a positive integer, $(n-6)$ is therefore positive and is an integer because n is an integer. The possible solutions are

$$\begin{array}{llllll} n = 7, & 8, & 9, & 10, & 12, & 18 \\ v = 14, & 8, & 6, & 5. & 4, & 3. \end{array}$$

The sides of the region are to be joined in pairs and so n cannot be odd. If $n = 8$ S is of order 8 and $T = S^4$. Then $ST = S^5$ and is of order 8. Since this is the value of v it follows that *there exists a map of a single octagon on a surface of genus two.*

If $n = 10$ we have $S^{10} = 1$, $T = S^5$, and $ST = S^6$ is of order 5 which is the value of the corresponding v. Hence, *there exists a map of a single decagon on a surface of genus two.*

If $n = 12$ we have $S^{12} = 1$, $T = S^6$, and $ST = S^7$ is of order 12. This is not the proper value for v, and hence there is no map. If $n = 18$ we have $S^{18} = 1$, $T = S^9$, and $ST = S^{10}$ is of order 9. There is no map in this case.*

If the map contains two regions we get in the same manner as above $(n-4)(v-2) = 8$. The possible solutions are

* The last two groups give maps on surfaces of genera 3 and 4 respectively.

$$n = 5, \quad 6, \quad 8, \quad 12$$
$$v = 10, \quad 6, \quad 4, \quad 3.$$

The dual of the map of a single decagon above corresponds to the case $n = 5$. Thus, *there exists a map of two pentagons on a surface of genus two.* We note that the second map given above is self-dual and therefore we do not expect to find its dual again. Since the group of order 10 is either cyclic or dihedral and since the map corresponding to the latter group is on a sphere there is no other map of two pentagons on a surface of genus two.

If $n = 6$ we note that the group must be Abelian and S and T must be independent generators, for $TST = S^m$ where $m^2 \equiv 1$, mod 6. The only solutions are $m = 1$, or 5. The latter gives a dihedral group and the map lies on a sphere. Hence, S and T are permutable and ST is of order 6 which is the corresponding value of v in the table above. Therefore, *there exists a map of two hexagons on a surface of genus two. The map is self-dual.*

If $n = 8$ we have $TST = S^m$ where $m^2 \equiv 1$, mod 8. The solutions are $m = 1$, 3, 5, and 7. The last gives the dihedral group of order 16 and need not be considered. The first gives ST of order 8 which is not the proper value for v. For $m = 3$ we may take $S = (abcdefgh)$ and $T = (bd)(cg)(fh)$, whence $ST = (adeh)(bgfc)$ which is of the proper order. Hence, *there exists a map of two octagons on a surface of genus two.* If $m = 5$ we may take S to be as above in which case T will be $(bf)(dh)$; their product is $(afgdebch)$ which is not of the proper order. Hence, *there is just one map of two octagons.*

Finally, if $n = 12$ $TST = S^m$ where $m^2 \equiv 1$, mod 12. The possible solutions are $m = 1, 5, 7, 11$. The first and last are impossible as in the preceding case. For the other two cases we may take S to be $(abcdefghijkl)$ and T and T' to be $(bf)(ck)(ei)(hl)$ and $(bh)(dj)(fl)$ respectively. The orders of ST and ST' are 4 and 6 and therefore the groups give maps on surfaces of genera 3 and 4 respectively. Hence, *there exists no map of two dodecagons on a surface of genus two.*

We have now considered all the possibilities when k, the number of regions, is less than 3. We tabulate the remaining possibilities in the following list.

In making out this list we may proceed according to values of k as we have done for the cases $k = 1$ and 2. To see that k cannot be greater than 28 we may put the equation given by the Euler formula in the form $2/v + 2/n + 4/kn = 1$. Since $k > 28$ and $n \geqq 3$, $1/v + 1/n \geqq 10/21$. Neither v nor n can be less than 3 and hence both must be less than 7, and if one of them is 4 the other must be less than 5. Possible pairs of values for v, n are 3, 3; 3, 4; 3, 5; 3, 6; 4, 3; 4, 4; 5, 3; 6, 3. The corresponding

values of $n/2 - (n+v)/v$ are $-1/2$, $-1/4$, $-1/10$, 0, $-1/3$, 0, $-1/3$, 0. None of these values are positive as they must be to satisfy the Euler formula, since k is necessarily positive. Hence *there is no map of more than* 28 *regions on a surface of genus two.*

The columns headed r and g give respectively the number of vertices and the orders of the groups. The list contains just one of a pair of dual maps, the one given being the one with the smaller value of k, for example, the first map would contain 3 10-sided regions coming together 3 at a vertex and its dual which is not listed would be made up of 10 triangles coming together 10 at a vertex. The only exceptions we have made are in the cases where the dual map would contain 1 or 2 regions.

	k	n	v	r	g		k	n	v	r	g
1.	3	10	3	10	30	7.	6	8	3	16	48
2.	3	4	12	1	12	8.	6	3	18	1	18
3.	4	9	3	12	36	9.	8	5	4	10	40
4.	4	6	4	6	24	10.	8	3	12	2	24
5.	4	5	5	4	20	11.	12	7	3	28	84.
6.	4	4	8	2	16						

We may dispose immediately of nos. 2, 6, 8, and 10. Since there does not exist a map of one 12-sided region or one 18-sided region on a surface of genus two *there exists no map corresponding to no.* 2 *or no.* 8. There exists a map of two octagons and no map of two dodecagons, hence *there exists a map of* 4 *4-sided regions* and *there does not exist a map of* 8 *triangles on a surface of genus two.*

A brief consideration shows that there can exist no regular map of 4 pentagons corresponding to no. 5. No region can touch itself along an edge unless it touches itself along every edge, in which case the map would consist of a single region of an even number of sides. If a given region of a regular map touches another region more than once along an edge it must touch each of the regions on its boundary the same number of times and hence if the number of its edges is a prime it can touch but one other region. Such a map can contain but two regions. Hence *there exists no map of* 4 *pentagons on a surface of genus two.*

For the remaining possibilities we examine the substitution groups of the proper order and degree, all of which, with one exception, have been listed.*

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* These groups are to be found as follows:
Degrees 4-8, Miller, *American Journal of Mathematics*, Vol. 28 (1899), pp. 287-338;
Degree 10, Cole, *Quarterly Journal*, Vol. 27 (1894), pp. 39-50.
Degree 12, Miller, *Quarterly Journal*, Vol. 28 (1896), pp. 193-231.

The degree of the group may be taken to be either k or r, since r is the number of regions of the dual map. The exception is that of a possible group of order 48 and degree 6 or 16 corresponding to no. 7 of the list. It is evident that if such a map exists its group cannot be represented on symbols for regions, but may be represented on symbols for regions of the dual map. Thus we would seek a substitution group of order 48 and degree 16. The groups of degree 16 have not been listed.

We take up the remaining cases in order.

1. We seek a group of order 30 and degree 10. No such group exists and hence *there exists no map.*

3. There are five groups of order 36 and degree 12.

The first two groups contain $(abcdef\cdot ghijkl)_{18}$ which is simply isomorphic with $(\alpha\beta\gamma)$ all $(\delta\epsilon\zeta)$ cyc. G_{18} thus contains operators of orders 2, 3, and 6. The two groups are obtained by combining G_{18} with $T_1 = (ag\cdot bh\cdot ci\cdot dj\cdot ek\cdot fl)$ and $T_2 = (aj\cdot bl\cdot ck\cdot dg\cdot ei\cdot fh)$ respectively. T_1 is permutable with every operator of G_{18}, but T_2 is permutable with G_{18} without being permutable with every operator. Neither of these groups contains an operator of order 9, for, since $T_1ST_1 = S$ and $T_2ST_2 = S'$ for every S where S and S' are in G_{18}, $(T_1S)^9 = 1$ would imply $T_1S^9 = 1$ and $(T_2S)^9 = 1$ would imply $T_2S(S'S)^4 = 1$ which are impossible since neither T_1 nor T_2 is in G_{18}.

The third and fourth groups may be managed in the same manner. The third is $\{(abcdef)$ cyc $(ghijkl)$ cyc$\}$ pos and $R_1 = (ajdg\cdot bkeh\cdot clfi)$ whose square is in G_{18}. The fourth is $\{(abcdef)_6\ (ghijkl)_6\}$ pos and $R_2 = (ajdg\cdot blfh\cdot ckei)$ whose square is also in the corresponding G_{18}. The first G_{18} contains operators of orders 2, 3, and 6; the second contains operators of orders 2 and 3 since $(abcdef)_6$ is simply isomorphic with the symmetric group of degree 3 and the products of the type $(ad)(bf)(ce)(ghi)(jkl)$ are not in G_{18}. An operator of order 9 would have to be outside G_{18} in either case and hence of the form RS where R is either R_1 or R_2 and S is in the corresponding G_{18}. Since $RSR = R^4\cdot RSR = R^2R^3SR = R^2S'' = S'$ where S'' and S' are in G_{18}, $(RS)^9 = 1$ would imply $RS(S'S)^4 = 1$ which is impossible since R is not in G_{18}.

The fifth group is $(abcd\cdot efgh\cdot ijkl)$ pos which is simply isomorphic with the alternating group of degree 4, and $(aei\cdot bfj\cdot cgk\cdot dhl)$ which is permutable with every operator of the G_{12}. G_{12} contains operators of orders 2 and 3 and G_{36} contains operators of orders 2, 3, and 6. Hence, *there exists no map of twelve triangles on a surface of genus two.*

4. We seek a group of order 24 and degree 6. There are three transitive groups of the proper order and degree. The first two $(\pm\, abcdef)_{24}$ and $(+\, abcdef)_{24}$, are simply isomorphic with the symmetric group of degree 4 and so contain no operators of order 6. Hence, neither of these can be the group of a regular map of six 4-sided regions coming together six at a vertex.

The third group is $(abcdef)_{24_6}$ which is simply isomorphic with the direct product of the alternating group on four letters and an operator of order two. Its operators are of orders 2, 3, and 6 and so it could not be the group of a map of 4-sided regions.

7. We seek a map of 16 triangles coming together 8 at a vertex. The dual map is composed of six octagons coming together three at a vertex. Since in the dual map $n > k$ each octagon would have to touch just four or two others and it would be impossible to represent the group on symbols for regions. Therefore we have no hope of finding the group among those of order 48 and degree 6. The groups of order 48 and degree 16 have not been listed. However, the group generated by S and T which satisfy the relations

$$S^8 = 1, \quad T^2 = 1, \quad (ST)^3 = 1, \quad \text{and} \quad (S^4T)^2 = 1$$

is of order 48.* To see this we note that S,

$$S_1 = TST, \quad S_2 = S^{-1}S_1S = S^6T, \quad S_3 = S^{-1}S_2S = S^5TS,$$
$$S_4 = S^{-1}S_3S = TS^6, \quad \text{and} \quad S_5 = TS_3T = S^3$$

constitute a complete conjugate set, S transforming S_1, S_2, S_3, S_4 cyclically and leaving S_5 fixed and T transforming S, S_2, and S_3 into S_1, S_4, and S_5 respectively. S represented on symbols for its conjugates is of order four, T is of order two, and ST is of order three transforming S, S_1, and S_4 cyclically and S_2, S_5, and S_3 likewise cyclically. The group represented on symbols for conjugates of S is the group of the cube, or the octahedral group, being generated by two operators of orders two and three whose product is of order four. Hence, the group generated by S and T is of order 48. Therefore, *there exists a map of six octagons coming together three at a vertex on a surface of genus two.*

9. We seek a group of order 40 and degree 8 or 10. There is one group of order 40 and degree 10, viz. $(abcde \cdot fghij)_{20}$ and $T = (af \cdot bg \cdot ch \cdot di \cdot ej)$ which is permutable with every element of the G_{20}. The G_{20} is simply isomorphic with the metacyclic group of degree 5. The elements of G_{20} are of

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* This group is among the groups of genus two given by Burnside. It is also the group described at the end of § 2.

orders 2, 4, and 5 and the elements outside of G_{20} are of orders 2, 4, and 10. The elements of order 5 with identity constitute a cyclic subgroup which is invariant under G_{20} and so under G_{40}. Let S and Σ be elements of orders 4 and 5 respectively of G_{20}. G_{40} is generated by S and an element of order two outside of G_{20} which must be of the form $T\cdot\Sigma^{-k}S^2\Sigma^k = T'$, for the group $[S, T']$ contains $S^2T' = TS^2\Sigma^{-k}S^2\Sigma^k = T\cdot\Sigma^{-k}$, since $S^{-1}\Sigma S = \Sigma^2$. $T\cdot\Sigma^{-k}$ is of order 10 and its square is a power of Σ, hence Σ is in $[S, T']$. The order of ST' is not 5 since all the operators of order 5 are in G_{20}. Hence, the map corresponding to such a choice of generators would not lie on a surface of genus two.* The generators might be chosen to be $S' = ST$ and $T'' = \Sigma^{-k}S^2\Sigma^k$ the first being any operator of order four outside of G_{20} and the second being in G_{20}. The above argument shows also that their product is not of order 5. Noting that there exists no group of order 40 and degree 8, we have the result that *there exists no regular map of ten 4-sided regions on a surface of genus two.*

11. This case is readily disposed of by observing that there exists no group of order 84 and degree 12.

Collecting our results we see that there exist just 8 regular maps on a surface of genus two. They are made up of

1 octagon, a self-dual map;
1 decagon and the dual map of
2 pentagons;
2 hexagons, a self-dual map;
2 octagons and the dual map of
4 quadrangles;
6 octagons and the dual map of
16 triangles.

* For neither choice of generators could ST be of order two since G_{40} is not dihedral; ST must therefore be of order four and the map lies on an anchor ring. See *Regular Maps on an Anchor Ring*, p. 234.

The Groups Belonging to a Linear Associative Algebra.

By Arthur Ranum

1. The present paper may be considered a generalization and extension of the ideas contained in a paper on "The Group-Membership of Singular Matrices" in the *American Journal of Mathematics,* vol. 31 (1909), pp. 18-41; but no knowledge of the earlier paper is here presupposed. Whereas there the algebra treated was that of all n-ary matrices whose elements belong to the field of all complex numbers, here we shall take a more general viewpoint and consider any linear associative algebra whatever in any field whatever.

Frequent references will be made to L. E. Dickson's *Algebras and Their Arithmetics,* University of Chicago Press (1923), and to G. Scorza's *Corpi Numerici e Algebre,* Messina (1921).

2. Groups and semi-groups * will be defined as follows. Given a set S of elements and a law of combination of the elements, under which they satisfy the associative law, then if the result of combining any two of the elements of S is an element of S, S is called a *semi-group.* If in addition S contains a unique identical element (identity) and for every element a unique inverse, then S is called a *group.*

Any algebra (and indeed any linear set) is evidently a group under addition, and any associative algebra is a semi-group, but not a group, under multiplication. It may be possible, however, to select certain sets of elements of an associative algebra, each of which is a group under multiplication. The present investigation is concerned with precisely these groups and with the relations existing between them and the remaining elements (if any), namely the non-group-members. We assume, therefore, that every algebra considered is associative.

The only case heretofore studied is that in which the algebra A has a modulus m, and the group concerned has m for its identical element and so consists either of the totality G of those elements x of A that have inverses x^{-1} such that $xx^{-1} = m$, or of the elements of a subgroup of G.

3. But it is evident that whether the algebra A has a modulus or not, any element u of A for which $u^2 = u$, that is any idempotent element or zero,

* See H. Hilton's *Theory of Groups of a Finite Order* (1908), p. 51.

can be chosen as the identity of a group belonging to A; and conversely, any group belonging to A must have for its identity either an idempotent element or zero. Moreover, if u is any idempotent element, Scorza (Art. 160, p. 266) shows that an element x of A will satisfy the condition $ux = xu = x$, if and only if, it belongs to the sub-algebra uAu, having the modulus u. Hence any group having u for its identity must belong to uAu.

We shall use the term *entire group* for the largest such group G_u, of which all the others are sub-groups; G_u is then the totality of the elements x of uAu having inverses x^{-1} in uAu such that $xx^{-1} = u$; moreover $x^{-1}x = u$, and x uniquely determines x^{-1}, in uAu and therefore also in G_u. Now this is equivalent, as proved by Scorza (II, Art. 109), to saying that an element x of the sub-algebra uAu will belong to the group G_u, if and only if, it is neither zero nor a divisor of zero in uAu (it may be a divisor of zero in A). Among these divisors of zero there may, of course, be other idempotents and therefore other groups.

It is clear that the element zero is the identity of an entire group G_0, to which no other element of A belongs; that is, $G_0 = 0$. Since a nilpotent algebra contains no idempotents, it contains no groups except G_0; all its other elements are non-group-members.

We shall use small Greek letters to denote scalars, or numbers of the field F in which the algebra is defined, and small Roman letters to denote elements of the algebra. It is clear that if x is any *group-member*, belonging to the group G_u, then αx $(\alpha \neq 0)$ also belongs to G_u; its inverse is $(1/\alpha)x^{-1}$. Moreover, every element αu, where u is idempotent and $\alpha \neq 0$, is obviously invariant in the group G_u.

If the identity u of an entire group G_u is a principal or a primitive idempotent of A, we shall call G_u a *principal* or a *primitive* group of A, respectively. Hence, if u is any idempotent whatever, G_u is the principal group of the sub-algebra uAu.

4. We define the *left* (*right*) *characteristic* (Scorza II, Art. 104) of an element x of an algebra A to be the order of the linear set $xA\,(Ax)$. It may be mentioned in passing that when we introduce the left (right) matrix or linear transformation corresponding to x, or in Dickson's nomenclature (§ 58), the first (second) matrix of x, then the left (right) characteristic of x is equal to the *rank* of its left (right) matrix (Scorza II, Art. 193).

THEOREM: *All the elements of a group G_u belonging to an algebra A have the same left characteristic and the same right characteristic; and if x and y are any two such elements, then*

$$xA = yA = uA, \qquad Ax = Ay = Au, \tag{1}$$

and therefore

$$xAy = uAu. \tag{2}$$

Proof: Let z be the element of G_u for which $xz = y$, $x = yz^{-1}$. Then since $A \geqq zA$, $xA \geqq xzA = yA$; and since $A \geqq z^{-1}A$, $yA \geqq yz^{-1}A = xA$; hence $xA = yA$, and the rest of the theorem follows immediately.

Corollary: If x is any group-member belonging to A, then

$$xA = x^2A \quad (Ax = Ax^2). \tag{3}$$

5. THEOREM: *Any two entire groups are mutually exclusive; they cannot have any element in common.*

Proof: It will evidently be sufficient to show that if two entire groups G_{u_1} and G_{u_2} have an element x in common, their idempotents u_1 and u_2 coincide. Since $xu_1 = x = xu_2$, therefore

$$x(u_1 - u_2) = 0. \tag{4}$$

Let B denote the sub-algebra xA; in view of (3), we have

$$xB = B. \tag{5}$$

Since (1) shows that $xA = u_1A = u_2A$, we see that $xA (= B)$ contains the elements u_1, u_2, and therefore $u_1 - u_2$. But it is known (Dickson, p. 30, lemma 3, and p. 45, 2nd paragraph) that in view of (5), any equation of the form $xz = y$, where y is an element of B, has a unique solution, in B, for z. Hence (4) implies that $u_1 = u_2$, and the theorem is proved.

6. THEOREM: *If the algebra A has an invariant sub-algebra B, any group G_u belonging to A either has no element in common with B or belongs entirely to B.* That is, an invariant sub-algebra either pays no attention to a group or swallows it whole.

Proof: Let x be an element common to G_u and B. Since B is invariant, the identical element $u = xx^{-1}$ of G_u will belong to B. For the same reason, *any* element $y = uy$ of G_u will belong to B. That is, G_u belongs entirely to B.

Now consider an algebra A of index r (Dickson, § 29), so that

$$A > A^2 > \cdots\cdots > A^r = A^{r+1} = \cdots$$

Let x be any group-member contained in A, and G_u the entire group to which it belongs. Then x^r will belong to G_u and also to the sub-algebra A^r. The

latter, however, is obviously invariant in A, and so, by the last theorem, contains the entire group G_u and therefore the element x.

Hence *every group of an algebra A of index r belongs wholly to its subagebra A^r*; that is, every element belonging to A but not to A^r is a non-group-member.

7. Since a division algebra has a modulus m and no divisors of zero, all of its elements are group-members and the entire algebra is exhausted by two groups, namely G_0 and the principal group G_m. The question arises as to whether there exist other algebras that are exhausted by their group-members. The answer is given by the

THEOREM: *A necessary and sufficient condition that an algebra A consist entirely of group-members is that A be either a division-algebra or a direct sum of division-algebras.* If t is the number of division-algebras in A, then 2^t is the number of entire groups exhausting the alegbra.

Proof: Let A consist entirely of group-members. Then it contains no nilpotent elements and therefore no exceptional (maximal nilpotent invariant) sub-algebra. Hence (Dickson, §§ 37 and 40) it must be semi-simple, and either A or each of its irreducible components A_i must be a simple algebra with a modulus. But by Wedderburn's theorem (Dickson, § 51) A_i is a direct product of a division-algebra B_i and a complete matric algebra * C_i, each of which is a sub-algebra of A_i and therefore of A. Since in our case C_i has no nilpotent elements, it must be of order 1, and $A_i = B_i$ which is a division-algebra.

Conversely, let A be the direct sum of the division algebras A_i $(i = 1, \cdots, t)$, and let u_i be the modulus of A_i. Let x be any element of A; then $x = \sum_1^t x_i$, where x_i belongs to A_i. If every term $x_i = 0$, x is the group-member 0; if at least one term $x_i \neq 0$, there is no loss of generality in writing $x = \sum_1^s x_i$, where $s \leq t$ and $x_i \neq 0$ $(i = 1, \cdots, s)$. If $u = \sum_1^s u_i$ and $y = \sum_1^s x_i^{-1}$, where x_i^{-1} is the inverse of x_i in the principal group of A_i, we see at once, in view of the equations $A_iA_j = 0$ $(i \neq j)$, that $u^2 = u$, $xu = ux = x$, and $xy = u$. Hence x belongs to the entire group G_u. The number of entire groups in A is evidently 2^t.

* I prefer not to use Dickson's term "simple matric algebra" for C_i, because other matric algebras, sub-algebras of C_i, may also be simple.

COROLLARY 1: If the number of groups in an algebra is infinite, or is finite, but not a power of 2, then the groups cannot exhaust the algebra.

Since the proof we have given of the first part of the above theorem is based entirely on the absence of nilpotent elements from the algbra, we have

COROLLARY 2: If an algebra contains non-group-members, some of them, at least, must be nilpotent.

Since an algebra is nilpotent, if and only if, it has no idempotents (Dickson, § 31), we have the

THEOREM: *An algebra will contain no group-members except zero, if and only if, it is nilpotent.*

8. We shall now derive some criteria for determining whether a given element x of an algebra A is a group-member or not, and if it is, we shall show how to find the entire group to which it belongs. The corollary of § 4 gives a necessary condition for group-membership, which is not, however, sufficient in all cases.

In the first place it is known (Scorza, II, Arts. 105-109) that when A has a modulus, then x will belong to the principal group of A, if and only if, the left (right) characteristic of x is equal to the order of A, that is, if

$$xA = A \quad (Ax = A).$$

THEOREM: *If an algebra A has a modulus or merely a right (left) modulus, any element x of A will be a group-member or not, according as the left (right) characteristic of x^2 is equal to, or less than, the left (right) characteristic of x, that is, according as*

(6) $$x^2A \gtreqless xA \quad (Ax^2 \gtreqless Ax).$$

By a *right modulus* (Scorza, II, Arts. 67, 70) we mean an element v such that $xv = x$, for every x in A. For a *left modulus,* similarly, we have $vx = x$. A modulus is therefore both a right modulus and a left modulus.

Proof: Let A have a right modulus v, and let x be any element of A such that $x^2A = xA$. The sub-algebra $xA = B$ will contain $xv = x$. Since $xB = B$, any equation

(7) $$xy = z,$$

where z belongs to B, will have a unique solution, in B, for y exactly as in § 5. Putting $z = x$, we see that B contains a unique element u such that

(8) $$xu = x.$$

Hence $xuz = xz$, $x(uz - z) = 0$, where z is *any* element of B. But this last equation is another special case of (7), and therefore implies

$$uz = z, \tag{9}$$

so that u is a left modulus of B. Putting $z = x$ and also $z = u$, in (9), and combining the results with (8), we have $u^2 = u$, $ux = xu = x$. Hence u is idempotent and x belongs to the sub-algebra uAu, by § 3.

Finally, putting $z = u$ in (7), we see that B contains a unique element x^{-1}, such that $xx^{-1} = u$. Hence $xx^{-1}x = ux = xu$, $x(x^{-1}x - u) = 0$, which implies $x^{-1}x = u$. Moreover, x^{-1} evidently belongs not merely to B but also to its sub-algebra $Bu = xAu = uAu$. Hence x is a group-member and belongs to the group G_u. For the case where A has a left modulus and $Ax^2 = Ax$, the proof is exactly similar. In view of the corollary of § 4, the theorem is completely established.

9. In order to find a criterion applicable to the general case where A is any algebra, not necessarily possessing either right or left modulus, we make use of an *augmented algebra* A' (Dickson, p. 97), defined as follows: Letting $A = (u_1, \cdots, u_n)$, where $u_1, \cdots, u_n$ form a set of basic units of A, we put

$$A' = (u_0, u_1, \cdots, u_n) = A + (u_0), \tag{10}$$

where u_0 is an annexed unit such that

$$u_0^2 = u_0, \quad u_0 u_i = u_i = u_i u_0 \quad (i = 1, \cdots, n).$$

Evidently A' is an associative algebra as well as A, and it has the modulus u_0. Since A is an invariant sub-algebra of A', it follows (§ 6) that every entire group of A is an entire group of A'. By the last theorem, an element x of A is a group-member or not, according as $x^2A' \gtreqless xA'$.

In order to express this condition in terms of the original algebra A, we make use of (10) and the derived equations $xA' = xA + (x)$, $x^2A' = x^2A + (x^2)$, and so obtain the criterion

$$x^2A + (x^2) \gtreqless xA + (x), \tag{11}$$

which can, however, be simplified and written in the form

$$x^2A \gtreqless xA + (x). \tag{12}$$

For, let (11_1) be satisfied,* so that x belongs to a group, say to G_u. Then xA

* By (11_1) and (11_2) we shall mean the formula (11) taken with the upper and lower signs, respectively.

contains $xu = x$, and $xA + (x) = xA$. Hence (12_1) reduces to (3) and is therefore satisfied. On the other hand, if (11_2) is satisfied, so that x is not a group-member, then since in every case $x^2A \gtreqless x^2A + (x^2)$, (12_2) is satisfied a fortiori. We have now proved the

THEOREM: *An element x of any algebra A is a group-member or not, according as it satisfies* (12_1) *or* (12_2).

$xA + (x)$ is obviously a sub-algebra of A. If A has a right modulus, (12) reduces to (6), as it should.

10. If x is an element of an algebra A, it is obvious (Scorza, II, Art. 113) that there always exists a positive integer r, such that

$$(13) \qquad A > xA > x^2A > \cdots > x^{r-1}A > x^rA = x^{r+1}A = \cdots,$$

except that for $r = 1$, $A \gtreqless xA = x^2A = \cdots$, so that the first inequality of (13) may possibly be replaced by an equality. If $r > 1$, this means that the left characteristics of $x, \cdots, x^r$ form a steadily decreasing set of integers, while the left characteristics of $x^r, x^{r+1}, \cdots$, are all equal.

We shall call r the *left index* of x in A. The *right index* is defined in a similar manner. If s is the least positive integer (if any) for which x^s is a group-member, we shall call s the *group-index* of x. Hence the group-index of a group-member is equal to 1.

If A has a right modulus and if x is an element of A whose left-index is r, then by the theorem of § 8, x^t is a group-member or not, according as $x^tA = x^{2t}A$ or $x^tA > x^{2t}A$, that is, in view of (13), according as $t \gtreqless r$ or $t < r$; hence r is the group-index of x. This gives us the

THEOREM: *If the algebra A has a right* (*left*) *modulus, the left* (*right*) *index of every element of A is equal to its group-index, and if A has a modulus, the left index, right index, and group-index of every element are all three equal.*

11. Now let A be any algebra whatever and A' its augmented algebra, as in § 9. Any element x of A will have the same group-index r in A' as in A, but not necessarily the same left-index. Since A' has a modulus, the theorem just proved shows that in A' the left-index of x is equal to its group-index r. Hence

$$(14) \qquad x^{r-1}A' > x^rA' = x^{r+1}A'$$

is a necessary and sufficient condition that x be of group-index $r > 1$. By means of the equations $x^rA' = x^rA + (x^r)$, etc., derived from (10), the condition (14) is expressible in terms of the original algebra A in the form

$$(15) \qquad x^{r-1}A + (x^{r-1}) > x^r A + (x^r) = x^{r+1}A + (x^{r+1}).$$

Finally we shall prove that (15) is equivalent to the simpler, though less symmetric, condition

$$(16) \qquad x^{r-1}A + (x^{r-1}) > x^r A + (x^r) = x^{r+1}A.$$

For, if x satisfies (15), x^r is a group-member and is therefore contained in $x^r A$. Hence x^{r+1} is contained in $x^{r+1}A$, and (15) reduces to (16) and indeed to the still simpler form

$$(17) \qquad x^{r-1}A + (x^{r-1}) > x^r A = x^{r+1}A,$$

which is a necessary, but not a sufficient, condition.

Conversely, if x satisfies (16),

$$(18) \qquad x^r A + (x^r) \leqq x^{r+1}A + (x^{r+1}).$$

But since the left member of (18) always contains the right member, the two members are equal, and (15) is satisfied. We have now established the

THEOREM: *In any algebra A an element x is of group-index $r > 1$, if and only if, it satisfies the condition* (16).

If A has a right modulus, (16) reduces to (13), in agreement with the theorem of § 10.

12. THEOREM: *If in any algebra the group-index of an element x is r, its left (right) index must be either r or $r - 1$.*

Proof: Assuming (14) and noting that by (17) $x^r A = x^{r+1}A$, we see that it will be sufficient to show that

$$x^{r-2}A > x^{r-1}A \geqq x^r A,$$

or in other words that

$$(19) \qquad x^{r-2}A > x^r A,$$

where $r > 2$; if $r = 2$, we write A in place of $x^{r-2}A$. Now $x^{r-1} = x^{r-2}x$ is an element of $x^{r-2}A$, but is not an element of $x^r A$. For if it were, it would be an element of $x^r A'$, and $(x^{r-1}) \leqq x^r A'$ would imply $x^{r-1}A' \leqq x^r (A')^2 = x^r A'$, which would contradict (14). This proves the theorem.

Incidentally, it follows that *every element x of an algebra has a finite group-index,* and *if x^r belongs to an entire group G, every higher power of x will also belong to G.*

13. *Definition:* The totality of the elements of an entire group G belonging to an algebra A, together with all those non-group-members, if any,

each of which has some power (with positive integral exponent) belonging to G, we shall call a *pseudo-group* $\bar{G}$, associated with G.

Obviously a pseudo-group G has the property that any positive integral power of an element of $\bar{G}$ is an element of $\bar{G}$; notice, however, that the product of two elements of $\bar{G}$ is not necessarily an element of $\bar{G}$, so that a pseudo-group is not necessarily a semi-group, much less a group.

But *every commutative pseudo-group is a semi-group.* For let r and s $(r \gtreqless s)$ be the group-indices of any two elements x and y of the commutative pseudo-group $\bar{G}$ associated with the group G. Then $(xy)^r = x^r \cdot y^r$; and since x^r and y^r both belong to G, their product belongs to G, and xy belongs to $\bar{G}$.

Evidently a pseudo-group $\bar{G}$ is a group, if and only if, all its elements are of group-index 1, so that $\bar{G} = G$.

No two pseudo-groups can have an element x in common. For if they did, x^r, for a sufficiently high value of r, would belong to two distinct entire groups, which is impossible.

Thus the pseudo-groups of an algebra, like the entire groups, are mutually exclusive; but unlike the entire groups, they exhaust the algebra, as we see by reference to § 12. Hence *every element of an algebra belongs to one and only one pseudo-group.*

A nilpotent algebra clearly consists of just one pseudo-group, which is also a semi-group. If an algebra is not nilpotent, its nilpotent elements (if any) together with zero, constitute a pseudo-group, which is a semi-group, provided all its elements are properly nilpotent or zero (Dickson, § 32).

14. Examples.

(a) Let $A = (x, y)$, with the multiplication table

	x	y
x	x	y
y	0	0

Every element $u = x + \beta y$ is idempotent, and since $uAu = (u)$, G_u consists of the elements $\alpha(x + \beta y)$, where $\alpha \neq 0$. Besides the entire groups G_u, one for each value of β, and the group $G_0 = 0$, there is one pseudo-group $\bar{G}_0$ not a group, whose elements βy are nilpotent or zero.

(b) Let $A = (x, y, z)$, with the modulus $x + y$ and the multiplication table

	x	y	z
x	x	0	0
y	0	y	z
z	0	z	0

There are three idempotents: $x + y = u$, x, and y. Hence there are four entire groups: G_u, G_x, G_y, and G_0; and A is made up of four pseudo-groups $\bar{G}_u$, $\bar{G}_x$, $\bar{G}_y$, and $\bar{G}_0$. Let $v = \alpha x + \beta y + \gamma z$.

(1) If $\alpha\beta \neq 0$, v belongs to $\bar{G}_u = G_u$, the principal group.

(2) If $\beta = 0$, $\alpha \neq 0$, v belongs to $\bar{G}_x$, and if $\gamma = 0$, to G_x; if $\gamma \neq 0$, v is of group-index 2 and $v^2 = \alpha^2 x$ belongs to G_x.

(3) If $\alpha = 0$, $\beta \neq 0$, v belongs to $\bar{G}_y = G_y$.

(4) If $\alpha = \beta = 0$; v belongs to $\bar{G}_0$; if $\gamma \neq 0$, v is of group index 2 and $v^2 = 0$.

15. We shall now derive another criterion of group-membership, one that involves the minimum equation (Dickson, § 68) satisfied by an element x of an algebra A. In the first place it is known (Scorza, II, Arts. 117-125) that the minimum equation $\phi(\omega) = 0$ of x will have a constant term $\alpha \neq 0$, if and only if, A has a modulus m and x belongs to the principal group of A. In the corresponding equation $\phi(x) = 0$ the constant term α is to be replaced by αm.

Now consider any algebra A and any element x of A whose minimum equation $\phi(\omega) = 0$ lacks a constant term and is therefore of the form

$$\omega^s + \cdots + \gamma\omega^2 + \beta\omega = 0, \tag{20}$$

so that

$$x^s + \cdots + \gamma x^2 + \beta x = 0. \tag{21}$$

Suppose that x is a group-member, belonging to a group G_u, and that x^{-1} is its inverse, so that $xx^{-1} = u$. Then equation (21), multiplied through by x^{-1}, becomes

$$x^{s-1} + \cdots + \gamma x + \beta u = 0. \tag{22}$$

It follows that $\beta \neq 0$; for if it were zero, (22) shows that x would satisfy an equation of lower degree than its minimum equation (20).

Conversely, suppose that $\beta \neq 0$. In the augmented algebra A' (§ 9), with a modulus u_0, x will satisfy the same minimum equation (20) as in A. But in A' equation (21) can be written $x = x^2 y$, where

$$y = -(1/\beta)(x^{s-2} + \cdots + \gamma u_0).$$

Since $yA' \gtreqless A'$, it follows that $xA' = x^2 yA' \gtreqless x^2 A'$. But $xA' \gtreqless x^2 A'$, in every case; hence in this case $xA' = x^2 A'$, from which we conclude (§ 8) that x is a group-member.

Combining these results, we have the

THEOREM: *In any algebra A an element x, whose minimum equation is*

$\phi(\omega)=0$, will be a group-member, if and only if, $\phi(\omega)$ is not divisible by ω^2; the case in which $\phi(\omega)$ is not even divisible by ω will occur only when A has a modulus and x belongs to the principal group of A.

Given a group-member x and its minimum equation (20), we can find the idempotent of the entire group to which it belongs, in view of (22), by the formula

$$u=-(1/\beta)\,(x^{s-1}+\cdots+\gamma x); \tag{23}$$

and we can then obviously find the inverse of x by the formula

$$x^{-1}=-(1/\beta)\,(x^{s-2}+\cdots+\gamma u).$$

Moreover, the potential * algebra $B=(x, x^2, \cdots, x^{s-1})$, which is a commutative sub-algebra of A, and also of uAu, contains u, by (23), and u is its modulus. Hence if G_u is the entire group, in A, to which x belongs, and if H_u is the principal group of B, to which x also belongs, then H_u is an Abelian sub-group of G_u.

16. THEOREM: *In an algebra A a non-group-member x, whose minimum equation is $\phi(\omega)=0$, will be of group-index $r(>1)$, if and only if, $\phi(\omega)$ is divisible by ω^r but not by ω^{r+1}.*

Proof: Let the minimum equation of x be

$$\omega^s+\cdots+\beta\omega^{r+1}+\alpha\omega^r=0 \qquad (r>1,\ \alpha\neq 0). \tag{24}$$

Then in the augmented algebra A', with a modulus u_0, we have $x^r=x^{r+1}y$, where

$$y=-(1/\alpha)\,(x^{s-r-1}+\cdots+\beta u_0);$$

and exactly as in § 15 we see that $x^rA'=x^{r+1}A'$, so that x^r is a group-member, by (14). But x^{r-1} is not a group-member. For if it were, *its* minimum equation $\psi(\omega)=0$, by the last theorem, would not be divisible by ω^2, so that x would satisfy an equation of the form

$$\omega^{t(r-1)}+\cdots+\beta_1\omega^{r-1}+\alpha_1=0 \qquad (\alpha_1,\ \beta_1 \text{ not both}=0),$$

and the left member of this equation would have to be divisible (Scorza, II, Art. 125) by the left member of (24), which is impossible.

Hence r is the group-index of x. The converse is obvious and the theorem is proved.

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* See Scorza, II, Art. 117.

The potential sub-algebra

$$B = (x, \cdots, x^r, \cdots, x^{s-1})$$

is evidently of index r, and

$$B^r = (x^r, \cdots, x^{s-1}).$$

It is easy to prove that B^r has a modulus u, which is the idempotent of the entire group to which x^r belongs. By § 6, the group-members of B are all contained in B^r.

17. Let us now briefly consider an algebra A in a field F that is non-modular and therefore infinite. A member x of a group G_u belonging to A will usually be of infinite period (order), but may be of finite period m. In that case $x^m = u$ and $x^{m+1} - x = 0$, so that x satisfies the equation $\omega(\omega^m - 1) = 0$. Hence if $\phi(\omega) = 0$ is the minimum equation of x, $\phi(\omega)$ must divide $\omega(\omega^m - 1)$.

When we speak of the roots of an equation $\psi(\omega) = 0$, we shall understand that for the moment the field F has been extended sufficiently to make $\psi(\omega)$ completely reducible (resolvable into linear factors).

Since in a non-modular field the roots of the equation $\omega(\omega^m - 1) = 0$ are all distinct (simple), the roots of $\phi(\omega) = 0$ are also distinct and are roots of unity or zero.

Conversely, if x has a minimum equation $\phi(\omega) = 0$ whose roots are distinct and are roots of unity or zero, then by § 15 x belongs to a group, say G_u, and satisfies an equation of the form $\omega(\omega^m - 1) = 0$, so that $x^{m+1} = x$. This equation, multiplied by by x^{-1}, gives $x^m = u$. Hence we have the

THEOREM: *In order that an element of an algebra in non-modular field may be a group-member of finite period, it is necessary and sufficient that its minimum equation have distinct roots, which are roots of unity or zero.*

18. If m and r are the least positive integers for which the equation $x^{m+r} = x^r$ holds, then x is clearly of group-index r and its higher powers x^r, $x^{r+1}, \cdots, x^{m+r-1}$ form a cyclic group of order m; and conversely. We state without proof the

THEOREM: *An element of an algebra in a non-modular field will be a non-group-member, whose higher powers form a cyclic group, if and only if, its minimum equation is of the form $\omega^r\psi(\omega) = 0$, where $r > 1$ and the roots of $\psi(\omega) = 0$ are distinct roots of unity.*

19. Returning to the general case of any field, we shall now examine

more in detail the groups contained in an algebra, and show how their structure is related to the structure of the algebra.

If $x \neq 0$ and $y \neq 0$, while $xy = yx = 0$, we shall say that x and y are *mutually nilfacient.*

Suppose that the algebra A is the *direct sum* of two sub-algebras A_1 and A_2, $A = A_1 \oplus A_2$. The equations $A_1 A_2 = A_2 A_1 = 0$ express the fact that any two non-zero elements of A, one from each component, are mutually nilfacient. Let G_1 and G_2 be any two groups contained in A_1 and A_2, respectively, and let u_1 and u_2 be their respective identities.

By the equation $G = G_1 + G_2$ we shall mean that G consists of all the elements obtained by adding an element of G_1 to an element of G_2. G is evidently a group and its identity is $u_1 + u_2$. Conversely, any group whatever of A can be expressed uniquely in the form $G = G_1 + G_2$.

Now assume that neither G_1 nor G_2 is zero, so that G belongs to neither A_2 nor A_1. Notice that G_1 and G_2 are not sub-groups of G; they have no elements in common with G or with each other. Put

$$H_1 = G_1 + u_2,$$
$$H_2 = u_1 + G_2.$$

Then H_1 and H_2 are sub-groups of G having no common elements except the identity $u_1 + u_2$. Obviously they are simply isomorphic with G_1 and G_2, respectively. Moreover every element of H_1 is commutative with every element of H_2, and $H_1 H_2 = G_1 + G_2 = G$. Hence G is the *direct product* of H_1 and H_2. By an easy generalization we arrive at the

THEOREM: *If an algebra A is the direct sum of two or more sub-algebras, any group belonging to A, but not to one of its components, is the direct product of two or more sub-groups.*

Hence the groups considered in the first theorem of § 7 are direct products.

20. If B is an invariant proper sub-algebra of A, consider the difference algebra $A' = A - B$, whose elements are the classes $[x]$ of elements of A, taken modulo B. If $u^2 = u$ and $xy = u$, then $[u]^2 = [u]$ and $[x][y] = [u]$. Hence if x runs through the elements of a group G belonging to A, $[x]$ will run through the elements of a group G' belonging to A'. In particular, if x lies in B, G lies wholly in B, by § 6, and $G' = [0]$.

On the other hand, if G' is a given group belonging to A', it seems difficult to determine whether there exists a group G belonging to A and having

one or more elements in each class $[x]$ of G'. It is clear, however, that the set of *all* the elements of A belonging to the various classes of G' is a semi-group.

21. In the special case where B is nilpotent, we can get more precise results.

THEOREM: *If the algebra A has a modulus u and a nilpotent invariant sub-algebra E, and if G' is the principal group of the difference algebra $A' = A - E$, then the totality of the elements of A belonging to the various classes $[x]$ of G' is a group G, namely the principal group of A.*

Proof: It is clear that $[u]$ is the modulus of A'. Let $[x]$ be any element of G' and $[y]$ its inverse, so that $[x][y] = [u]$. Let x and y be *any* elements of A belonging to the classes $[x]$ and $[y]$ respectively. Then since $xy \equiv u \pmod{E}$, we have

$$xy = u - e, \tag{25}$$

where e is an element of E. E being nilpotent, $e^r = 0$, for a sufficiently high value of r. Since u and e are commutative,

$$(u - e)(u^{r-1} + eu^{r-2} + \cdots + e^{r-1}) = u^r - e^r = u;$$

that is, $(u - e)z = u$, where

$$z = u + e + e^2 + \cdots + e^{r-1},$$

and by (25), $xyz = u$. Hence x belongs to the principal group G of A, and its inverse is yz. The converse follows from § 20, and the theorem is proved.

COROLLARY: *The elements $u + e$ of the class $[u]$ constitute an invariant sub-group $u + E$ of G.*

For the equation $x^{-1}ux = u$ implies $[x^{-1}][u][x] = [u]$, whence

$$x^{-1}(u + e)x = u + e'.$$

Evidently G' is the quotient-group of G with respect to $u + E$. Hence if H' is any sub-group of G', the elements of A belonging to the classes of H' form a sub-group H of G containing $u + E$, and $H' = H/(u + E)$.

22. The following theorem will not be needed later and its proof will therefore be omitted.

THEOREM: *Let A be a non-nilpotent algebra having a nilpotent invariant sub-algebra E; let G' be any entire group $\neq [0]$ of the difference algebra*

$A - E$, and $[u]$ its idempotent. Then there exists in A at least one entire group G whose idempotent u lies in the class $[u]$, and if $B = uAu$, the elements of G are precisely those elements of B that belong to the various classes $[x]$ of G'.

If $B \wedge E = F$, so that F is a nilpotent invariant sub-algebra of B, and H' is the principal group of $B - F$, then by § 21 G can also be described as the set of elements of B that belong to the classes of H'.

23. We are now prepared to study the structure of any entire group G_u whatever in any algebra A whatever. The trivial case where $G_u = 0$, and therefore also the case where A is nilpotent, will be excluded.

If A is not semi-simple, we denote its exceptional (maximal nilpotent invariant) sub-algebra by E, and put $A' = A - E$; then A' is the direct sum of simple algebras $A'_1, \cdots, A'_t$, or if $t = 1$, is itself simple. The case in which A is semi-simple can be regarded as included by putting $E = 0$, $A' = A$ and $A'_i = A_i$ $(i = 1, \cdots, t)$.

The idempotent u and the group G_u are said to have the signature *

$$(p_1, \cdots, p_t),$$

provided $u = u_1 + \cdots + u_t$ $(u_i u_j = 0,\ i \neq j)$ and therefore

$$[u] = [u_1] + \cdots + [u_t],$$

where $[u_1]$ is an idempotent or zero of A_i' $(i = 1, \cdots, t)$, and is either expressible as the sum of p_i primitive idempotents, mutually nilfacient, or is itself primitive $(p_i = 1)$ or finally is zero $(p_i = 0)$. At least one of the integers p_i must be positive; if u is a principal idempotent, they are all positive (Dickson, p. 88), and the signature of u is said to be the *signature of the algebra A*.

Suppose that the number of positive integers p_i is r $(0 < r \leqq t)$; if $r < t$, we can rearrange the irreducible components A_i' of A' and write the signature of G_u in the form

$$(p_1, \cdots, p_r, 0, \cdots, 0), \tag{26}$$

so that $u = u_1 + \cdots + u_r$.

We know by § 3 that G_u is the principal group of the sub-algebra $A_u = uAu$, which has a modulus u. Moreover, A_u has the signature

$$(p_1, \cdots, p_r). \tag{27}$$

Hence our problem of finding the structure of an entire group of signature

* Scorza, II, Arts. 249-252.

(26) in an algebra A reduces to a study of the principal group G_u of a sub-algebra A_u of signature (27). If $r > 1$, we shall show that G_u is expressible, in a certain sense, in terms of groups for which $r = 1$. The latter groups will be studied in §§ 31-34.

24. Continuing the notation of the last section, we now proceed to decompose the algebra A_u, employing Dickson's method (§ 57), except that his u is a principal idempotent of A, while ours is any idempotent for which $r > 1$. We put

$$(28) \qquad A_{ij} = u_i A u_j, \qquad E_{ij} = u_i E u_j \qquad (i, j = 1, \cdots, r).$$

Let E_u be zero or the exceptional sub-algebra of A_u; then

$$(29) \qquad A_u = \Sigma A_{ij}, \qquad E_u = \Sigma E_{ij}, \\ A_{ij} = E_{ij} \quad (i \neq j),$$

and E_{ii} is zero or the exceptional sub-algebra of A_{ii}. Let

$$(30) \qquad A'_u = A_u - E_u, \qquad A'_{ii} = A_{ii} - E_{ii} \qquad (i = 1, \cdots, r);$$

then $A'_u = \Sigma A'_{ii}$, which is a direct sum. Let the principal groups of the various algebras

$$A_u, \quad A_{ii}, \quad A'_u, \quad A'_{ii}$$

be denoted by

$$G_u, \quad G_{ii}, \quad G'_u, \quad G'_{ii},$$

respectively. Then by § 19, $G'_u = \Sigma G'_{ii}$.

Now let x be any element of A_u; then $x = \Sigma x_{ij}$, where x_{ij} is in A_{ij}. Let $[x]$ be the corresponding element of A'_u, that is, a class of elements of A_u, taken modulo E_u. Then $[x] = \Sigma [x_{ii}]$, since $[x_{ij}] = [0]$, is $i \neq j$; and $[x_{ii}]$ is an element of A'_{ii}, that is, a class of elements of A_{ii}, taken modulo E_{ii}.

By the theorem of § 21, x will belong to the group G_u, if and only if, $[x]$ belongs to the group G'_u, which implies that $[x_{ii}]$ belongs to the group G'_{ii} $(i = 1, \cdots, r)$, and conversely. But by applying the same theorem (§ 21) to the algebra A_{ii}, we see that $[x_{ii}]$ will belong to G'_{ii}, if and only if, x_{ii} belongs to the principal group G_{ii} of A_{ii}. Notice that no condition is imposed on x_{ij} $(i \neq j)$, which is therefore free to roam anywhere in the algebra A_{ij} (if $A_{ij} \neq 0$); whereas x_{ii} must have an inverse x_{ii}^{-1} in the algebra A_{ii}, such that $x_{ii} x_{ii}^{-1} = u_i$.

We have now proved the

THEOREM: *Let G_u be an entire group of signature $(p_1, \cdots, p_r, 0, \cdots, 0)$ in an algebra A and let its idempotent be $u = u_1 + \cdots + u_r$ $(r > 1;$ $u_i u_j = 0, i \neq j)$, where u_i is of signature $(0, \cdots, 0, p_i, 0, \cdots, 0)$; also let $A_{ij} = u_i A u_j$, so that $uAu = \Sigma A_{ij}$; then*

$$G_u = \Sigma G_{ii} + \Sigma' A_{ij}, \tag{31}$$

where G_{ii} is the principal group of the sub-algebra A_{ii} $(i = 1, \cdots, r)$, and where the symbol Σ' means that $i \neq j$.

Since the linear sets A_{ij} are supplementary in their sum, a term A_{ij} of (31) has no element except zero in common with the sum of the remaining terms, and a term G_{ii} has no element whatever in common with the sum of the remaining terms.

We have therefore expressed any group G_u in terms of the groups G_{ii}, each of which is the principal group of an algebra A_{ii} of signature (p_i); either A_{ii} is a simple algebra (if $E_{ii} = 0$) or its difference algebra $A_{ii} - E_{ii}$ is simple.

25. Let us now consider some of the more striking sub-groups of the entire group G_u. First of all it is clear that $\bar{G}_u = \Sigma G_{ii}$ is a sub-group, and since the sub-algebra ΣA_{ii}, of which $\bar{G}_u$ is the principal group, is the direct sum of $A_{11}, \cdots, A_{rr}$, it follows (§ 19) that G_u is the direct product of the sub-groups

$$\begin{aligned} G_1 &= G_{11} + u_2 + \cdots + u_r, \\ G_2 &= u_1 + G_{22} + \cdots + u_r, \\ &\cdots\cdots\cdots\cdots \\ G_r &= u_1 + u_2 + \cdots + G_{rr}. \end{aligned} \tag{32}$$

In the special case where $\Sigma' A_{ij} (= \Sigma' E_{ij}) = 0$ and therefore in the still more special case where A_u is semi-simple, the sub-group $\bar{G}_u$ is the entire group G_u.

By (29_2) and the corollary of § 21 we see that

$$u + E_u = \Sigma(u_i + E_{ii}) + \Sigma' E_{ij}$$

is an invariant sub-group of G_u, and that the quotient-group $G_u/(u + E_u)$ is $G'_u = \Sigma G'_{ii}$. By the same corollary, $u_i + E_{ii}$ is an invariant sub-group of G_{ii}. The greatest common sub-group of $\bar{G}_u$ and $u + E_u$ is $J = \Sigma(u_i + E_{ii})$, which is invariant in $\bar{G}_u$, but not usually in $u + E_u$ or in G_u. J, like H, is obviously a direct product of r sub-groups $J_1, \cdots, J_r$, say, unless some of the linear sets E_{ii} are zero. If all of them are zero, so that ΣA_{ii} is semi-simple, then $J = u$.

It is easy to see that G_u is generated by its two sub-groups $\bar{G}_u$ and $u + E_u$ and that every element x of G_u is expressible, in more than one way unless $J = u$, as a product yz, where y and z are elements of $\bar{G}_u$ and $u + E_u$, respectively, and also as a product $z'y'$. Hence $\bar{G}_u$ and $u + E_u$ are permutable * groups.

Other notable sub-groups of G_u are of the form

$$\sum_1^r G_{ii} + \Sigma' A_{kl},$$

where k and l do not run through *all* the values from 1 to r, but only a part of them, e. g., from 1 to s $(s < r)$. Still others are of the form

$$(\Sigma G_{ii} + \Sigma' A_{ij}) + (\Sigma G_{kk} + \Sigma' A_{kl}),$$

where i, j run through a part of the values from 1 to r and k, l the remaining values, e. g., $i, j = 1, \cdots, s$, and $k, l = s + 1, \cdots, r$. The latter sub-group is evidently the direct product of the groups

$$\Sigma G_{ii} + \Sigma' A_{ij} + u_{s+1} + \cdots + u_r,$$
$$u_1 + \cdots + u_s + \Sigma G_{kk} + \Sigma' A_{kl}.$$

Similarly there are sub-groups expressible as the sum of three or more terms, each of the form $\Sigma G_{ii} + \Sigma' A_{ij}$.

26. In any field for which the principal theorem on algebras (Dickson, § 78) has been proved, that is, in the present state of our knowledge, in any non-modular field, the structure of the groups contained in an algebra can be determined somewhat more completely than we have been able to do so far.

Let A be an algebra in a non-modular field, and G_u an entire group belonging to A. As in §§ 23, 24, let E_u be the exceptional sub-algebra (if it exists) of $A_u = uAu$, and let $A'_u = A_u - E_u$. Then by the principal theorem on algebras there exists in A_u a semi-simple sub-algebra B_u equivalent to A'_u, and having one and only one element in each class $[x]$ of elements of A_u, modulo E_u, so that $A_u = B_u + E_u$.

Moreover, B_u has the same modulus u as A_u; hence the principal group H_u of B_u is a sub-group of the principal group G_u of A_u. But H_u has one and only one element in each class $[x]$ of elements of G_u, modulo E_u. Hence by the theorem of § 21,

(33) $$G_u = H_u + E_u.$$

* Burnside, *Theory of Groups*, § 33. The definition is there given only for finite groups, but is clearly applicable also to infinite groups.

By § 25, G_u has the invariant sub-group $u + E_u$.

27. *Definition:* If a group G has two sub-groups G_1 and G_2, having only the identity in common, and if every element of G is expressible in one and only one way as a product $g_1 g_2$, where g_1 and g_2 are elements of G_1 and G_2, respectively, and also as a product $g'_2 g'_1$, then we shall call G the *product* of G_1 and G_2. The two sub-groups are therefore permutable. Every direct product is a product, but not conversely.

In a similar manner we define the product of more than two groups; each group will then have only the identity in common with the product of all the rest.

We shall now prove that G_u (§ 26) *is the product of its sub-groups* H_u *and* $u + E_u$.

For by (33) any element of G_u can be written $h + e$, where h and e are elements of H_u and E_u, respectively. Our goal will be reached if we can find unique elements e', e'' in E_u, such that

$$h(u + e') = h + e = (u + e'')h. \tag{34}$$

But (34_1) implies that $he' = e$, and therefore that $e' = h^{-1}e$, which is in E_u, since the latter is invariant in A_u. Conversely, $e' = h^{-1}e$ implies (34_1). Similarly we can find e'' uniquely in E_u.

28. For the case in which G_u is of signature (26), where $r > 1$, let us see what further light is shed by our new result on the analysis of G_u given in §§ 24, 25. In the first place every algebra A_{ii} $(i = 1, \cdots, r)$ will now have a simple sub-algebra B_{ii} equivalent to A'_{ii}, such that $A_{ii} = B_{ii} + E_{ii}$ and $B_u = \Sigma B_{ii}$, which is a direct sum. Hence if H_{ii} is the principal group of B_{ii}, we have

$$H_u = \Sigma H_{ii},$$

and by the theorem of § 19, H_u is the direct product of the r sub-groups

$$\begin{aligned} H_1 &= H_{11} + u_2 + \cdots + u_r, \\ &\cdots\cdots\cdots\cdots \\ H_r &= u_1 + u_2 + \cdots + H_{rr}. \end{aligned} \tag{35}$$

Combining this result with that of § 27, we have the

Theorem: *Any group* G_u *of signature* $(p_1, \cdots, p_r, 0, \cdots, 0)$, *where* $r > 1$, *that belongs to an algebra* A *in a non-modular field, can be expressed*

as a product of $r+1$ sub-groups $H_1, \cdots, H_r, u+E_u$, each of which has only the identity u in common with the product of all the rest; $u+E_u$ is invariant in G_u and H_i $(i=1, \cdots, r)$ is simply isomorphic with the principal group of a simple algebra B_{ii}; the product of $H_1, \cdots, H_r$ is their direct product.

29. The sub-group $\bar{G}_u = \Sigma G_{ii}$ (§ 25) can now be further decomposed. For since $A_{ii} = B_{ii} + E_{ii}$, $G_{ii} = H_{ii} + E_{ii}$, and this group, by § 27, is the product of H_{ii} and $u_i + E_{ii}$. Hence, comparing (35) and (32), we see that G_i is the product of H_i and J_i (§ 25), where

$$J_1 = (u_1 + E_{11}) + u_2 + \cdots + u_r,$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$J_r = u_1 + u_2 + \cdots + (u_r + E_{rr}).$$

Finally $\bar{G}_u$ is the product of H_u and J, where H_u is the direct product of $H_1, \cdots, H_r$ and J is the direct product of $J_1, \cdots, J_r$. That is, *$\bar{G}_u$ is the product of the $2r$ sub-groups H_i, J_i $(i=1, \cdots, r)$.*

Another sub-group worth mentioning is $H_1 + E_u = H_{11} + u_2 + \cdots + u_r + \Sigma E_{ij}$, which is evidently the product of H_1 and $u + E_u$.

30. *Example.* Let $A = (u_1, u_2, x, y)$, with the modulus $u = u_1 + u_2$ and the multiplication table

	u_1	u_2	x	y
u_1	u_1	0	x	0
u_2	0	u_2	0	y
x	x	0	0	0
y	y	0	0	0

Its principal group G_u is of signature $(1, 1)$. Since $A_{11} = (u_1, x)$, $A_{12} = 0$, $A_{21} = (y)$, and $A_{22} = (u_2)$, the groups G_{11}, G_{22}, $\bar{G}_u$, and $G_u = G_{11} + A_{12} + A_{21} + G_{22}$ consist of the respective sets of elements $\alpha u_1 + \gamma x$ $(\alpha \neq 0)$, $\beta u_2 (\beta \neq 0)$, $(\alpha u_1 + \gamma x) + \beta u_2 (\alpha\beta \neq 0)$, and $\alpha u_1 + \beta u_2 + \gamma x + \delta y (\alpha\beta \neq 0)$.

A has the exceptional sub-algebra $E_u = (x, y)$ and the semi-simple sub-algebra $B = (u_1, u_2)$. Hence G_u is the product of the three sub-groups H_1, H_2 and $u + E_u$, whose respective sets of elements are $\alpha u_1 + u_2$ $(\alpha \neq 0)$, $u_1 + \beta u_2$ $(\beta \neq 0)$, and $u + \gamma x + \delta y$. By § 21, $u + (x)$ and $u + (y)$ are invariant sub-groups of G_u; indeed their product is $u + E_u$.

A has a variety of other entire groups besides G_u, but we shall not go into that.

31. Returning to the general case of an algebra A in an arbitrary field, let us now examine a group G_u of signature $(p_1, 0, \cdots, 0)$, which is the principal group of a sub-algebra uAu of signature (p_1). The groups G_{ii} (§ 24), in terms of which we found the more complicated groups to be expressible, are all of this type. In order to simplify the notation, we shall confine ourselves to the sub-algebra uAu, denoting it by A and its signature by (p).

It is known (Scorza, II, Art. 266; Dickson, § 56) that A is the direct product of a complete matric sub-algebra M of order p^2 and a sub-algebra B (of order r, say) having a primitive modulus; hence $A = B \times M$ is of order rp^2. A, B and M all have the same modulus u. If E is zero or the exceptional sub-algebra of B, $E \times M$ is zero or the exceptional sub-algebra of A. Since $B - E$ is a division algebra, the only divisors of zero in B (if any) are nilpotent and belong to E. All the other elements of B (except zero) belong to its principal group.

The modulus $u = u_1 + \cdots + u_p$ can be expressed, in many ways, as a sum of p primitive idempotents $u_1, \cdots, u_p$, which are mutually nilfacient. Let the basic units of M be c_{ij} $(i, j = 1, \cdots, p)$, where $c_{ij}c_{jk} = c_{ik}$, $c_{ij}c_{lk} = 0$ $(j \neq l)$, and $c_{ii} = u_i$ $(i = 1, \cdots, p)$. Every element x of A (Scorza, II, p. 245) can then be written

$$x = \Sigma b_{ij}c_{ij},$$

where b_{ij} is an element (not necessarily a basic unit) of B. If we regard c_{ij} as a p-rowed matrix having all its elements zero except the one in the ith row and jth column, which is $= 1$, then

$$x = \begin{pmatrix} b_{11} & \cdots & b_{1p} \\ \cdot & \cdot \quad \cdot & \cdot \\ b_{p1} & \cdots & b_{pp} \end{pmatrix} = (b_{ij}), \text{ and } u = \Sigma u c_{ii} = (\delta_{ij} u),$$

where

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$

This new kind of matrix (Dickson, §§ 97, 98) involves a twofold non-commutativity (if B is a non-commutative algebra), first because if $x' = (b'_{ij})$, then $xx' \neq x'x$, and second because if $xx' = x'' = (b''_{ij})$, so that

$$b''_{ik} = \sum_j b_{ij}b'_{jk}, \quad \text{then} \quad b_{ij}b'_{jk} \neq b'_{jk}b_{ij}.$$

32. Let G, H and K be the principal groups * of A, B and M, respectively; H and K are obviously sub-groups of G. In order to obtain a precise determination of the group G, it became necessary to invent a new kind of determinant that will be explained in a later paper.

At present we shall confine ourselves to the special case where B is a commutative algebra; the ordinary kind of determinant will answer our purpose, although its elements are not, as usual, scalars, but elements of B.

THEOREM: *If B is a commutative algebra, an element (b_{ij}) of the algebra $A = B \times M$ will belong to the principal group G of A, if and only if, the determinant $|b_{ij}|$ is neither nilpotent nor zero; that is, if $|b_{ij}|$ belongs to the principal group H of B.*

For, if (b_{ij}) belongs to G, it must have an inverse (b'_{ij}), such that $(b_{ij})(b'_{ij}) = (\delta_{ij}u)$, whence

$$|b_{ij}| \cdot |b'_{ij}| = u^p = u.$$

Therefore $|b_{ij}|$ must belong to H. The converse is evidently true and the theorem is proved.

If B is a division algebra $(E = 0)$, the criterion becomes simply

$$|b_{ij}| \neq 0.$$

33. Even when B is non-commutative, we can easily determine some of the more important sub-groups of G and express their elements as matrices. Take K, for instance. Since the elements of M are of the form $k = (\alpha_{ij}u)$, where α_{ij} is a scalar, the elements k of its principal group K are characterized by the condition $|\alpha_{ij}| \neq 0$.

Then take H. Its elements are evidently the matrices $h = (\delta_{ij}b)$, where b is any element of B that is neither nilpotent nor zero.

Since $hk = kh = (\alpha_{ij}b)$, the sub-groups H and K are permutable and generate a sub-group $\bar{G}$ consisting of the matrices $\bar{g} = (\alpha_{ij}b)$, where $|\alpha_{ij}| \neq 0$ and b is neither nilpotent nor zero. H and K are invariant in $\bar{G}$, but not necessarily in G. The greatest common sub-group J of H and K consists of the elements $(\delta_{ij}\alpha u)$, where $\alpha \neq 0$. J is invariant in G (and therefore in $\bar{G}$), because all its elements are invariant. The quotient-group

* Of course A has a multitude of other entire groups besides G, any one of which, G_1, is of signature (p_1), where $p_1 < p$. G_1 is itself the principal group of a sub-algebra of order rp_1^2 and can be represented either as a group of p_1-rowed matrices or as a group of "singular" p-rowed matrices, of which the constituents are in either case elements of B. See *American Journal of Mathematics*, Vol. 31 (1909), §§ 10, 11.

$\bar{G}/J$ consists of the classes $[\bar{g}]$, each class $[\bar{g}]$ consisting of all the scalar multiples $\alpha\bar{g}$ of $\bar{g}$ where $\alpha \neq 0$.

Since the analogous quotient-groups H/J and K/J have no elements in common except the identity $[u]$, it follows that G/J is the direct product of H/J and K/J. $\bar{G}$ may therefore be called the *quasi-direct product* of H and K. This gives us the

THEOREM: *In an algebra A having a modulus and the signature (p), which is the direct product of the sub-algebras B and M, the principal group of A has a sub-group $\bar{G}$ which is the quasi-direct product of the principal groups of B and M.*

The set of all the possible products of two factors, one from B and the other from M, or in other words the set of all the materices $(\alpha_{ij}b)$, where the α's and b are unrestricted, is a curious affair; for although it is clearly a semi-group, it is not a group, nor even a linear set, much less a sub-algebra.

Another sub-group of G is our old friend $u + EM$ (§ 21, corollary), which is invariant in G and consists of the matrices $(\delta_{ij}u + e_{ij})$, where the elements e_{ij} belong to E. Similarly $u + E$, consisting of the matrices $(\delta_{ij}(u + e))$, is an invariant sub-group of H.

34. In a non-modular field we can, as in §§ 26-29, analyse our group G of signature $(p, 0, \cdots, 0)$ more fully. Our algebra $A = B \times M$ will then contain a simple sub-algebra $A' = B' \times M$, equivalent to $A - EM$, such that $A = A' + EM$ and $B = B' + E$. Let G' and H' be the principal groups of A' and B', respectively; then $G = G' + EM$ and $H = H' + E$. The elements of H' are the matrices $(\delta_{ij}b')$, where b' is any element not zero of the division algebra B'; and the elements of G' are matrices of the form (b'_{ij}), where the elements b'_{ij} belong to B' and are such that if B' is a commutative algebra, $|b'_{ij}| \neq 0$. By § 27, *G is the product of its sub-groups G' and $u + EM$,* and H is the product of H' and $u + E$.

35. The groups studied in this paper are somewhat analogous to certain known groups arising in the theory of numbers. Consider the m classes $[x]$ of integers congruent to x with respect to a given composite modulus m. Since the set A of these classes is closed under addition and multiplication, it has many of the properties of a linear associative (and commutative) algebra. Hence it is not surprising that some of the classes form groups and others are non-group-members.

For example, let $m = p^2q$, where p and q are distinct primes. There are four entire groups G_m, G_{p^2}, G_q, G_0, and A is made up of four pseudo-groups G_m, G_{p^2}, $\bar{G}_q$, and $\bar{G}_0$.

(1) If x is prime to m, $[x]$ belongs to the principal group G_m, which is of order $p(p-1)(q-1)$.

(2) If $x \equiv 0 \pmod{q}$ and $\not\equiv 0 \pmod{p}$, $[x]$ belongs to G_{p^2}, of order $p(p-1)$.

(3) If $x \equiv 0 \pmod{p}$ and $\not\equiv 0 \pmod{q}$, $[x]$ belongs to $\bar{G}_q$, of order $p(q-1)$. If in addition $x \equiv 0 \pmod{p^2}$, $[x]$ belongs to G_q, of order $q-1$; while if $x \not\equiv 0 \pmod{p^2}$, $[x]$ is a non-group-member and $[x]^2$ belongs to G_q.

(4) If $x \equiv 0 \pmod{pq}$, $[x]$ belongs to $\bar{G}_0$, of order p. If $x \equiv 0 \pmod{p^2q}$, $[x] = [0] = G_0$; while if $x \not\equiv 0 \pmod{p^2q}$, $[x]$ is nilpotent and $[x]^2 = [0]$.

CORNELL UNIVERSITY,
June, 1926.

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